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New graph polynomials in parametric QED Feynman integrals

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ABSTRACT

In recent years enormous progress has been made in perturbative quantum field theory by applying methods of algebraic geometry to parametric Feynman integrals for scalar theories. The transition to gauge theories is complicated not only by the fact that their parametric integrand is much larger and more involved. It is, moreover, only implicitly given as the result of certain differential operators applied to the scalar integrand $\exp(-\Phi_\Gamma/\Psi_\Gamma)$, where Ψ_Γ and Φ_Γ are the Kirchhoff and Symanzik polynomials of the Feynman graph Γ. In the case of quantum electrodynamics we find that the full parametric integrand inherits a rich combinatorial structure from Ψ_Γ and Φ_Γ. In the end, it can be expressed explicitly as a sum over products of new types of graph polynomials which have a combinatoric interpretation via simple cycle subgraphs of Γ.

1. Introduction

The parametric version of Feynman integrals has long been an extremely useful tool in the study of perturbative quantum field theory [1–5]. Moreover, over the last decade a number of fascinating breakthroughs have unveiled deep connections to algebraic geometry and number theory, and have motivated mathematicians to study Feynman integrals, their periods, as well as connections to combinatorics and geometry [6–12]. However, most of this takes place in the realm of scalar quantum field theories due to the complications that the tensor structure of gauge theories brings to Feynman integrals. Not only does the parametric integrand in quantum electrodynamics (the simplest gauge theory) contain a number of (traces of) Dirac matrices, which we will discuss separately in future work. It also contains a complicated rational function in the Schwinger parameters, momenta, metric tensors etc. in front of the usual integrand of a scalar Feynman integral. While the rough structure of

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this function has been known for a long time[1] to be a certain sum over products of polynomials that are somehow related to derivatives of the second Symanzik polynomial, there has been no direct combinatorial interpretation of what these polynomials are. In this article we give such an interpretation of the form $\sum_C \pm \Psi_{\Gamma /\!\!/ C}$, where the sum is over certain cycle subgraphs of the Feynman graph Γ and $\Psi_{\Gamma /\!\!/ C}$ is its Kirchhoff polynomial after contraction of that cycle.

Moreover, we believe that combining the results of this article with a systematic combinatorial treatment of Dirac matrices in future work will allow us to give a version of a parametric QED Feynman integrand that is essentially a single scalar integrand and includes a number of intricate cancellations that reduce the size of the integrand at higher loop orders by several orders of magnitude compared to the naive version Eq. (62). A thus simplified Feynman integral will be much easier to handle, for example when trying to extend Brown's and Kreimer's parametric renormalisation procedure [13] (recently applied to great effect in [14]) to QED, and allow for a better understanding of non-scalar Feynman amplitudes. In particular, we see this as a first step in answering a number of long standing questions in QED, for example the cancellation of transcendental terms in the beta function [15–17]. Finally, since most of the combinatorics underlying our result is independent of the specific case of QED, it should be possible to generalise the insights gained in this article to the non-abelian case by studying the corolla differential of [18].

We begin by recapitulating some basic graph theory and the definitions of the Kirchhoff and Symanzik polynomials in Sections 2.1 and 2.2. For more detail we suggest the reader consult the excellent review article [19] or the classic book [20]. Building on that we define our new *cycle polynomials* and discuss a number of examples and properties in Section 2.3. In particular we would like to highlight the three identities proved in the lemmata 2.9–2.11, since they are the fundamental building blocks for the proof of our main result and also quite fascinating in their own right. After briefly introducing parametric Feynman integrals for non-experts and discussing the peculiarities of quantum electrodynamics in Section 3 we are ready to state and prove Theorem 4.1.

2. Graphs and graph polynomials

2.1. Graphs, subgraphs and Feynman graphs

A *graph* G is an ordered pair (V_G, E_G) of the set of *vertices* $V_G = \{v_1, \dots, v_{|V_G|}\}$ and the set of *edges* $E_G = \{e_1, \dots, e_{|E_G|}\}$, together with a map $\partial : E_G \to V_G \times V_G$, which is usually realised by drawing the graph in the plane. We will need our graphs to be directed, however as usual the particular choice of direction for each edge is arbitrary and will have no influence on the results of this article. For a directed edge $e \in E_G$ we write $\partial_-(e) \in V_G$ for its start vertex and $\partial_+(e) \in V_G$ for its target vertex, such that

$$\partial : e \mapsto (\partial_-(e), \partial_+(e)) \tag{1}$$

Unless explicitly stated otherwise we assume G to be connected, but its subgraphs may have multiple components. If a subgraph $g \subset G$ does not contain isolated vertices (which is the case for all the types of subgraphs we discuss below) it is uniquely defined by its edge set via $\partial(E_g)$ and we use the notation for the edge subset and the actual subgraph interchangeably.

2.1.1. Types of subgraphs

There are a multitude of significant types of graphs. For our purposes we concentrate on three of them, spanning trees, bonds and cycles.

A *spanning tree* $T \subset G$ is a tree (i.e. a connected and simply connected graph) that contains all vertices of G. In other words,

$$h_0(T) = 1 \qquad h_1(T) = 0 \qquad V_T = V_G$$

where h_i denotes the i-th Betti number of a graph. We denote the set of all spanning trees of G by $\mathcal{T}_G$.

[1] It follows very directly from the Leibniz rule of differentiation. See also [3].