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An example of interplay between Physics and Mathematics: Exact resolution of a new class of Riccati Equations

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HIGHLIGHTS

- A resolutive strategy for Liouville SU(2) dynamical problems is used to solve DREs.
- A new class of exactly solvable Riccati equations is reported.
- Many examples illustrate our method and provide explicit solutions of selected DREs.

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ABSTRACT

A novel recipe for exactly solving in finite terms a class of special differential Riccati equations is reported. Our procedure is entirely based on a successful resolution strategy quite recently applied to quantum dynamical time-dependent SU(2) problems. The general integral of exemplary differential Riccati equations, not previously considered in the specialized literature, is explicitly determined to illustrate both mathematical usefulness and easiness of applicability of our proposed treatment. The possibility of exploiting the general integral of a given differential Riccati equation to solve an SU(2) quantum dynamical problem, is succinctly pointed out.

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1. Introduction

Theoretically investigating the properties of systems in different contexts [1–3], such as for example classical and quantum physics [4–16], mathematics [17–25], biology [26,27], one is led to the consideration of the following non-linear non-autonomous first order differential equation

$$y'(x) = A(x)y^2(x) + B(x)y(x) + C(x). \tag{1}$$

Here y and x are the dependent and independent variables, respectively, and $y'(x)$ denotes the first derivative of y with respect to x. The intrinsic nature of the variables x and y depends of course on the problem under scrutiny. Such an equation is known as the scalar Differential Riccati Equation (DRE) and the three coefficient functions are supposed to be continuous in some interval $I \in \mathbb{R}$ [28–30]. To emphasize how wide is the range of different situations where such an equation emerges, it has been suggested that the DRE, as given by Eq. (1), should be included in the list of the named equations of mathematical physics [31, and references therein].

One should not be too surprised by such a central role played by the DRE. Indeed, in 1769 [28, and references therein], Euler demonstrated that every given DRE may be always transformed into a second-order linear, generally non-autonomous, differential equation and vice versa. Thus, since such differential equations represent very often the starting point for theoretically investigating both classical and quantum systems [32,33], in accordance with first principles [34–38] or as a result of an approximation path [31,39–41], it becomes easy to understand the systematic emergence of an associated DRE in the problems under scrutiny.

The circumstance that no method is available to construct a particular integral of a DRE and the fact that, as well, no general protocol exists leading to the unique explicit solution of a Cauchy problem relative to a linear second order non-autonomous differential equation (2nd-LDE), may be seen as two different faces of the same mathematical difficulty. Then, the intimate link existing between the two classes of mathematical problems assures the possibility of writing down the general integrals of both the linear and the related non-linear differential equations as soon as one is able to exactly solve one of them. Such an expectation has spurred the research reported in the present paper, where the knowledge of the exact quantum dynamics of some SU(2) problems is exploited to bring to light a new class of exactly solvable DRE. Here "new" means that, to the best of the present authors' knowledge, DREs possessing the particular relationship between the three coefficients, we are going to evidence in the next sections, do not appear in the most commonly used handbooks [29,30] and apparently are not present in more recent literature as well.

Our starting point is the quantum dynamics of a semiclassical generalized Rabi system, here defined as a spin 1/2 subjected to an arbitrary time-dependent magnetic field. To find the evolution operator U related to an SU(2)-symmetry dynamics is equivalent to solve a Cauchy problem for a homogeneous system of two coupled first-order differential equations in the two time-dependent complex unknown entries of U, which, in addition, must satisfy the usual unitarity condition. This mixed system may always be converted into a new Cauchy problem related to a second-order linear non-autonomous differential equation which, in turn, may be transformed into a Cauchy–Riccati problem.

Generally speaking, no procedure is known to find, in finite terms, the time-dependence of U when no special prescription is postulated on the time-dependence of the magnetic field. Quite recently, however, novel resolution strategies for exactly solving the quantum dynamics of the SU(2) problem under scrutiny has been reported [42–45]. The approach in Ref. [42], in particular, succeeds in singling out classes of exactly solvable time-dependent SU(2) Hamiltonian models, providing at the same time the successful protocol leading to the explicit construction of the relative evolution operators U.

In this paper we show how to turn the knowledge of the solution of such a physical problem into a recipe to solve a mathematical problem of general interest. Our procedure consists of two steps. First, we introduce a simple transformation by which the Cauchy problem for the determination of the two entries of U is turned into a Cauchy–Riccati problem. In this way we arrive at a special class E of DREs whose coefficients are directly linked to the three time-dependent components of the magnetic field acting upon the spin. At this point the knowledge of a particular solution of the quantum dynamics of the spin 1/2 enables us to write down, systematically and in a simple way, the explicit solution of

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