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Contact Hamiltonian mechanics

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ABSTRACT

In this work we introduce contact Hamiltonian mechanics, an extension of symplectic Hamiltonian mechanics, and show that it is a natural candidate for a geometric description of non-dissipative and dissipative systems. For this purpose we review in detail the major features of standard symplectic Hamiltonian dynamics and show that all of them can be generalized to the contact case.

1. Introduction

The Hamiltonian formulation of classical mechanics is a very useful tool for the description of mechanical systems due to its remarkable geometrical properties, and because it provides a natural way to extend the classical theory to the quantum context by means of standard quantization. However, this formulation exclusively describes isolated systems with reversible dynamics, while real systems are constantly in interaction with an environment that introduces the phenomena of dissipation and irreversibility. Therefore a major question is whether it is possible to construct a classical mechanical theory that not only contains all the advantages of the Hamiltonian formalism, but also takes into account the effects of the environment on the system.

Several programmes have been proposed for this purpose (see e.g. [1] for a recent review). For example, one can introduce *stochastic dynamics* to model the effect of fluctuations due to the environment on the system of interest. This leads to stochastic equations of the Langevin or Fokker–Planck

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type with diffusion terms [2,3]. A different although related approach is the *system-plus-reservoir* technique, in which the system of interest is coupled to an environment (usually modeled as a collection of harmonic oscillators). The system and the environment together are considered as an isolated Hamiltonian system and after averaging out the environmental degrees of freedom one obtains the equations of motion for the system of interest, including dissipative terms. This is the case for example of the Caldeira–Laggett formalism [4–6]. An alternative approach is to propose *effective Hamiltonians* with an explicit time dependence that reproduce the correct Newtonian equation, including the dissipative forces. A famous example is the Caldirola–Kanai (CK) model [7–9]. Another proposal based on a nonconservative action principle, allows for time-irreversible processes, such as dissipation, to be included at the level of the action [10,11]. Finally, a more geometrical attempt towards the description of dissipative systems is given by the so-called *bracket formulation* of dynamical systems [12]. Here one generalizes the standard Poisson bracket to a noncanonical Poisson bracket and exploits the algebraic properties of the latter to include dissipation. The literature on all these proposals is very extensive and it is not our purpose here to review them in detail. We refer the interested reader to the standard references cited above and references therein.

Here we discuss a new proposal which consists in extending the symplectic phase space of classical mechanics by adding an extra dimension, thus dealing with a contact manifold instead of a symplectic one. Contact geometry arises naturally in mechanics. First of all, in describing mechanical systems where the Hamiltonian function explicitly depends on time, one usually appeals to an extended phase space, the additional dimension being time, endowed with the Poincaré-Cartan 1-form, which defines a contact structure on the extended space [13–15]. Besides, the time-dependent Hamilton–Jacobi theory is naturally formulated in this extended phase space [16,17]. Furthermore, it has recently been argued that symmetries of the contact phase space can be relevant for a (non-canonical) quantization of nonlinear systems [18].

In this work we consider the phase space of any (time-independent) mechanical system (either non-dissipative or dissipative) to be a contact manifold, but we take a different route from previous works. In fact, there are two main differences between our proposal and the previous ones. First, we do not assume that the additional dimension is time, letting the additional dimension be represented by a non-trivial dynamical variable. Second, we derive the equations of motion for the system from *contact Hamiltonian dynamics*, which is the most natural extension of symplectic Hamiltonian dynamics [14].

Contact Hamiltonian dynamics has been used already in thermodynamics (both equilibrium and not [19–24]) and in the description of dissipative systems at the mesoscopic level [25]. Furthermore, it has been recently introduced in the study of mechanical systems exchanging energy with a reservoir [26,27]. However, a detailed analysis of the dynamics of mechanical systems and a thorough investigation of the analogy with standard symplectic mechanics have never been pursued before. We show that the advantages of contact Hamiltonian mechanics are that it includes within the same formalism both non-dissipative and dissipative systems, giving a precise prescription to distinguish between them, that it extends canonical transformations to contact transformations, thus offering more techniques to find the invariants of motion and to solve the dynamics, and that it leads to a contact version of the Hamilton–Jacobi equation. We argue that these additional properties play a similar role as their symplectic counterparts for dissipative systems.

The structure of the paper is as follows: in Section 2, in order to make the paper self-contained, we review the main aspects of the standard mechanics of non-dissipative systems, with emphasis on the symplectic geometry of the phase space and the Hamilton–Jacobi formulation. In Section 3 the same analysis is extended to the case of contact Hamiltonian systems and it is shown by some general examples that this formulation reproduces the correct equations of motion for mechanical systems with dissipative terms. Besides, an illustrative example (the damped parametric oscillator) is worked out in detail in this section in order to show the usefulness of our method. Section 4 is devoted to a summary of the results and to highlight future directions. In particular, we discuss a possible extension of our formalism to quantum systems. Finally, in Appendices A and B we provide respectively a derivation of the invariants of the damped parametric oscillator and a constructive proof of the equivalence between the contact Hamilton–Jacobi equation and the contact Hamiltonian dynamics.

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