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Markov stochasticity coordinates

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ABSTRACT

Markov dynamics constitute one of the most fundamental models of random motion between the states of a system of interest. Markov dynamics have diverse applications in many fields of science and engineering, and are particularly applicable in the context of random motion in networks. In this paper we present a two-dimensional gauging method of the randomness of Markov dynamics. The method – termed *Markov Stochasticity Coordinates* – is established, discussed, and exemplified. Also, the method is tweaked to quantify the stochasticity of the first-passage-times of Markov dynamics, and the socioeconomic equality and mobility in human societies.

1. Introduction

Networks are ubiquitous. Indeed, networks are omnipresent in science and engineering, as well as in our daily lives [1–6]. In turn, as our world is by large dynamic and stochastic, random motions that take place in networks are of the utmost importance [7–12]. The most fundamental model of such random motion is *Markov dynamics*, which is represented by *Markov matrices* [13–15]. For given Markov dynamics the associated Markov matrix specifies the transition probabilities between the different nodes of the underlying network.

Given a pair of probability distributions we can calculate their respective entropies and thus conclude which of the two is more random [16–18]. In this paper we explore an analogous Markovian question: given a pair of Markov dynamics how can we determine which of the two is more random? To answer this question we devise the method of *Markov Stochasticity Coordinates*, which will be established, discussed, and exemplifies hereinafter.

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The Markov Stochasticity Coordinates quantify the intrinsic randomness of Markov dynamics via a pair of *stochasticity gauges*. Each of the two stochasticity gauges takes values in the unit interval—with the zero lower bound manifesting minimal randomness, and with the unit upper bound manifesting maximal randomness. The two stochasticity gauges are coupled, and their joint pair takes values in a triangle whose corners are $(0,0)$, $(1,0)$, and $(1,1)$. In a sense, the triangle's x-axis manifests the dynamics' long-term stochasticity, and the triangle's y-axis manifests the dynamics' short-term stochasticity.

The motivation to introduce the notion of stochasticity gauges, and then to employ it as the foundation of the Markov Stochasticity Coordinates, came from the notion of socioeconomic *inequality indices* [19–21]. Economists and social scientists apply inequality indices to quantify the socioeconomic inequality of wealth and income distributions in human societies. Inequality indices take values in the unit interval—with the zero lower bound manifesting perfect equality, and with the unit upper bound manifesting perfect inequality. Perhaps the best known example of a socioeconomic inequality index is the *Gini index* [22].

The motivation to use a two-dimensional coordinates representation came from *Modern Portfolio Theory* [23–25]. This theory is the cornerstone of contemporary financial investing, it was pioneered by Nobel Laureate Harry Markowitz, and it uses a two-dimensional coordinates representation to quantify assets' yields: one axis manifests the mean of the yields, and the other manifests the variance of the yields. This representation is of major importance and effectiveness, as it provides a tangible visualization of the interplay between risk and return.

The organization of the paper is as follows. We begin with recalling the notion of socioeconomic inequality indices (Section 2), tweak it and devise the notion of *stochasticity indices* (Section 2), and present a collection of examples of stochasticity indices (Section 3). Then, after setting the underlying Markov-dynamics model (Section 4), we establish the method of *Markov Stochasticity Coordinates* (Section 5), discuss it (Section 6), and demonstrate its application (Section 7). Thereafter, tweaking the Markov Stochasticity Coordinates, we further demonstrate their application in the context of the *first passage times* of Markov dynamics (Section 8), and in the context of quantifying the socioeconomic *equality* and *mobility* in human societies (Section 9). The paper ends with a conclusion (Section 10), followed by an Appendix.

2. Inequality and stochasticity

Consider a human society consisting of n individuals, each with a non-negative wealth. Label the individuals $i = 1, \ldots, n$, denote by w_i the wealth of individual i, and set $\mathbf{w} = (w_1, \ldots, w_n)$ to be the corresponding vector of wealth values. Here and hereinafter we assume that the wealth values are not all identically zero, and hence at least one wealth value is positive. The distribution of wealth within the society has two socioeconomic extremes. One extreme is *perfect equality*: all individuals share a common wealth value. The other extreme is *perfect inequality*: one single individual possessing a positive wealth, and all other individuals possessing zero wealth. From a socioeconomic perspective these two extremes manifest, respectively, pure communism and absolute monarchy.

An *inequality index*, $I(\mathbf{w})$, is a functional that quantifies the degree of the society's socioeconomic inequality [19–21]. Specifically, the inequality index $I(\mathbf{w})$ takes values in the unit interval, and it satisfies the four following properties. (I) *Permutation invariance*: if the wealth vector $\mathbf{w}'$ is a permutation of the wealth vector $\mathbf{w}$ then $I\left(\mathbf{w}'\right) = I(\mathbf{w})$; indeed, the society's socioeconomic inequality should not depend on the particular order by which its members are labeled. (II) *Scale invariance*: if c is a positive constant then $I(c\mathbf{w}) = I(\mathbf{w})$; indeed, the society's socioeconomic inequality should not depend on the particular currency by which the wealth of its members is measured. (III) *Characterization of perfect equality*: the inequality index attains its lower bound, $I(\mathbf{w}) = 0$, if and only if the society is in the state of perfect equality, i.e. pure communism. (IV) *Characterization of perfect inequality*: the inequality index attains its upper bound, $I(\mathbf{w}) = 1$, if and only if the society is in the state of perfect inequality, i.e. absolute monarchy.

As the wealth values are non-negative and not all zero, the scale invariance property implies that the wealth vector can be *normalized* to a probability vector. Indeed, consider the society's overall wealth, $w := w_1 + \cdots + w_n$, and apply the positive scale constant $c = 1/w$. The wealth vector $\mathbf{w} =$

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