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Annals of Physics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Dressed coherent states in finite quantum systems: A cooperative game theory approach

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ARTICLE INFO

Article history:
Received 18 October 2016
Accepted 1 December 2016
Available online xxxx

Keywords:
Coherent states
Finite quantum systems

ABSTRACT

A quantum system with variables in $\mathbb{Z}(d)$ is considered. Coherent density matrices and coherent projectors of rank n are introduced, and their properties (e.g., the resolution of the identity) are discussed. Cooperative game theory and in particular the Shapley methodology, is used to renormalize coherent states, into a particular type of coherent density matrices (dressed coherent states). The Q-function of a Hermitian operator, is then renormalized into a physical analogue of the Shapley values. Both the Q-function and the Shapley values, are used to study the relocation of a Hamiltonian in phase space as the coupling constant varies, and its effect on the ground state of the system. The formalism is also generalized for any total set of states, for which we have no resolution of the identity. The dressing formalism leads to density matrices that resolve the identity, and makes them practically useful.

1. Introduction

Coherent states have been studied extensively in the literature [1–3]. They are an overcomplete set of states, and there are subsets of the full set of coherent states which are total sets (i.e., there is no state which is orthogonal to all states in the subset). We consider quantum systems with d-dimensional Hilbert space [4–6], in which case the number of coherent states is d^2 [7,8]. We show that ideas from cooperative game theory can provide a deeper insight to the overcompleteness of coherent states, and their linear dependence (lack of linear independence) which is related to it.

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http://dx.doi.org/10.1016/j.aop.2016.12.002

Cooperative game theory [9–12] adds 'corrections' to the individual contribution of a player, which reflect his contribution to coalitions (aggregations) of players. This gives the Shapley value, which shows the share of a player in the 'total worth' of the game, and which renormalizes (dresses) his lone contribution, by adding his contribution to all possible coalitions. The sum of all the Shapley values, is the total worth of the game.

In analogy to this, we add to the one-dimensional projector corresponding to a coherent state, other terms that reflect its role within spaces spanned by aggregations of coherent states. This gives a renormalized (dressed) coherent state, which is a mixed state described by a coherent density matrix. The sum of all coherent density matrices is the identity (resolution of the identity).

Coherent states are used to define the Q-function of a Hermitian operator θ. The dressed coherent states define a generalized Q-function, a physical analogue of the Shapley values. As an application of these ideas, we study the relocation of a Hamiltonian in phase space as a function of the coupling constant, and how this affects the ground state of the system. Such calculations on large quantum systems, can be used in the study of phase transitions. Location indices of Hermitian operators in phase space, and comonotonicity (or cohabitation) intervals of the coupling constant, are used to quantify this relocation. These calculations are performed with respect to either Q-function or Shapley values, and their relative merits are discussed.

Most of the paper uses the formalism with coherent states. But we also consider a total set of states, which are not coherent states and for which we have no resolution of the identity. The renormalization formalism in this case, leads to density matrices that resolve the identity and can be used in practical applications.

In order to develop this formalism, we introduce in Section 2.2 coherent density matrices. They are generalizations of coherent states (which are pure states) to mixed states. We then introduce in Section 2.3 coherent projectors of rank n, to spaces spanned by aggregations of n coherent states. The terms coherent density matrices and coherent projectors, reflect the fact that they resolve the identity, and that there is a closure property where under displacements they are transformed into other coherent density matrices and coherent projectors, correspondingly.

In Section 3 we present briefly some concepts from cooperative game theory, which are needed later. The presentation uses the standard language of cooperative game theory, but it also introduces some 'quantum terminology', because our aim is to transfer these ideas in a quantum context. In Section 4 we explain in a precise manner, the analogies between cooperative game theory, and aggregations of coherent states. Möbius transformations are used to identify overlaps and avoid double-counting: in cooperative game theory, they quantify the added value in a coalition; and in a quantum context they describe the double counting due to the overcompleteness of the coherent states.

In Section 5 we transfer the concept of Shapley values into a quantum context. We show that they are generalized Q-functions with respect to a particular set of coherent density matrices, which can be regarded as renormalized (dressed) versions of the 'bare' coherent projectors. The dressing formalism is related to the non-orthogonality and non-commutativity of the coherent projectors, as discussed in Section 5.1. The properties of these coherent density matrices are presented in Propositions 5.2 and 5.5.

In Section 6 the formalism is generalized to any total set of states. As an application we study in Section 7, the relocation of a Hamiltonian in phase space as the coupling constant varies, and its effect on the ground state of the system. We conclude is Section 8 with a discussion of our results.

2. Generalized coherence in finite quantum systems

2.1. Coherent projectors

We consider a quantum system with variables in $\mathbb{Z}(d)$, described by a d-dimensional Hilbert space $H(d)$. $|X; n\rangle$ is the basis of position states, and $|P; n\rangle$ the basis of momentum states (X and P in the notation are not variables, they simply indicate position and momentum states). They are related

Please cite this article in press as: A. Vourdas, Dressed coherent states in finite quantum systems: A cooperative game theory approach, Annals of Physics (2016), http://dx.doi.org/10.1016/j.aop.2016.12.002