Group theory, entropy and the third law of thermodynamics

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Curado *et al.* [Ann. Phys. **366** (2016) 22] have recently studied the axiomatic structure and the universality of a three-parameter trace-form entropy inspired by the group-theoretical structure. In this work, we study the group-theoretical entropy $S_{a,b,r}$ in the context of the third law of thermodynamics where the parameters $\{a, b, r\}$ are all independent. We show that this three-parameter entropy expression can simultaneously satisfy the third law of thermodynamics and the three Khinchin axioms, namely continuity, concavity and expansibility only when the parameter b is set to zero. In other words, it is thermodynamically valid only as a two-parameter generalization $S_{a,r}$. Moreover, the restriction set by the third law i.e., the condition $b = 0$, is important in the sense that the so obtained two-parameter group-theoretical entropy becomes extensive only when this condition is met. We also illustrate the interval of validity of the third law using the one-dimensional Ising model with no external field. Finally, we show that the $S_{a,r}$ is in the same universality class as that of the Kaniadakis entropy for $0 < r < 1$ while it has a distinct universality class in the interval $-1 < r < 0$.

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I. INTRODUCTION

Recently, there has been a great deal of progress in constructing generalized statistical mechanics through the applications of the information-theoretic entropies. To list some of them, one can cite Tsallis [1], Rényi [2], Kaniadakis [3] entropies. These generalized entropies aim to explain the non-equilibrium stationary metastable states through the deformation in the underlying entropic structure. Along this direction, many important applications were reported in the fields of generalized reaction rates [4–7], quantum information [8–14], plasma physics [15–17], high energy physics [18–20] and the rigid rotators in modelling the molecular structure [21, 22]. The common feature of these entropies is to yield inverse power law distributions through the entropy maximization [1, 3, 23, 24].

It is also worth noting the recent progress in the use of the fractional calculus based entropies such as Ubriaco [25] and Machado [26] entropies. These fractional entropies have been applied to the study of the financial time series and stock market index [26, 27], used for explaining the relation between DNA and the fractional Brownian motion [28], image splicing [29], and a new definition of the complexity metrics [30].

A very general novel approach, first initiated in Ref. [31], is to obtain generalized entropy expressions through the underlying group-theoretical structure [32–34]. The point of departure in this approach is the observation that the thermodynamically admissible entropies should satisfy a general composability property i.e. their values on a system composed by the union of two statistically independent subsystems A and B should depend on the entropies of the two subsystems *only* modulo some deformation parameters [32–34]. In other words, $S(A \cup B) = \Phi(S(A), S(B))$ where S denotes the entropy functional and $\Phi(x, y)$ represents a smooth function of two real variables x and y. Note that the choice $\Phi(x, y) = x + y$ yields the well-known additivity associated with the Boltzmann-Gibbs entropy for example. The function $\Phi(x, y)$ is then to satisfy the group properties, namely, symmetry ($\Phi(x, y) = \Phi(y, x)$), associativity ($\Phi(x, \Phi(y, z)) = \Phi(\Phi(x, y), z)$) and null-composability ($\Phi(x, 0) = x$) [32–34]. The symmetry requirement ensures the interchangeability of the subsystems, the null-composability implies no change in the thermodynamics of the composed system when it includes a system with zero entropy [32]. Finally, the associativity generalizes the notion of composability to more than two systems [32]. The choice of the function $\Phi(x, y)$ satisfying these three group properties determines the group structure underlying the entropy expressions one obtains. For example, the choice $\Phi(x, y) = x + y$ represents the additive group law while $\Phi(x, y) = x + y + axy$ (a being a real continuous parameter)

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