Annals of Physics 375 (2016) 1–15

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

The Wigner function in the relativistic quantum mechanics

K. Kowalski *, J. Rembieliński

Department of Theoretical Physics, University of Łódź, ul. Pomorska 149/153, 90-236 Łódź, Poland

HIGHLIGHTS

- We study the Wigner function for a quantum spinless relativistic particle.
- We discuss the relativistic Wigner function introduced by Zavialov and Malokostov.
- We introduce relativistic Wigner function based on the standard definition.
- We find analytic expressions for relativistic Wigner functions.

ARTICLE INFO

Article history:
Received 14 July 2016
Accepted 29 September 2016
Available online 4 October 2016

Keywords:
Quantum mechanics
Relativistic quantum mechanics
Wigner quasi-probability function

ABSTRACT

A detailed study is presented of the relativistic Wigner function for a quantum spinless particle evolving in time according to the Salpeter equation.

1. Introduction

The Wigner function also referred to as the Wigner quasi-probability function is one of the most important concepts of nonrelativistic quantum mechanics. Its applications range from nonequilibrium quantum mechanics, quantum optics, quantum chaos and quantum computing to classical optics and signal processing. As far as we are aware, in spite of the fact that the paper by Wigner was dated 1932 [1], the relativistic generalization of the Wigner function in the simplest case of the spinless particle was introduced by Zavialov and Malokostov only in 1999 [2]. The dynamics of that relativistic Wigner function was studied in recent papers [3,4] by Larkin and Filonov. Clearly, the

* Corresponding author.
E-mail address: kowalski@uni.lodz.pl (K. Kowalski).

http://dx.doi.org/10.1016/j.aop.2016.09.016

difficulties in extending the concept of the Wigner function to the relativistic domain sometimes considered as unattainable [5] are closely related with problems in finding the relativistic counterpart of the Schrödinger equation, that is constructing the relativistic quantum mechanics. In this work we discuss the advantages and limitations of the Wigner function introduced by Zavialov and Malokostov and analyze an alternative relativistic generalization of the Wigner function based on the standard nonrelativistic formula that was applied earlier in the case of the Dirac particle. Both of these approaches utilize the relativistic quantum dynamics described by the spinless Salpeter equation. The theory is illustrated by concrete examples of relativistic Wigner function for a spinless free particle.

2. The Zavialov–Malokostov Wigner function

2.1. *Definition of the Wigner function*

We now summarize the basic facts about the Wigner function introduced in Ref. [2]. The point of departure in [2] was the following form of the non-relativistic Wigner function

$$W(\boldsymbol{x},\boldsymbol{p},t)=\frac{1}{(2\pi)^3}\frac{1}{\hbar^6}\int d^3\boldsymbol{p}_1d^3\boldsymbol{p}_2\tilde{\phi}^*(\boldsymbol{p}_1,t)\tilde{\phi}(\boldsymbol{p}_2,t)\delta\left(\boldsymbol{p}-\frac{1}{2}(\boldsymbol{p}_1+\boldsymbol{p}_2)\right)e^{\frac{i(\boldsymbol{p}_2-\boldsymbol{p}_1)\cdot\boldsymbol{x}}{\hbar}}, \tag{2.1}$$

where $\tilde{\phi}(\boldsymbol{p},t)$ is the Fourier transform of the wave function $\phi(\boldsymbol{x},t)$, that is

$$\tilde{\phi}(\boldsymbol{p},t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3\boldsymbol{x}\,e^{-i\frac{\boldsymbol{p}\cdot\boldsymbol{x}}{\hbar}}\phi(\boldsymbol{x},t). \tag{2.2}$$

Furthermore, Zavialov and Malokostov restrict to the case of the free relativistic evolution described by the Salpeter equation (see [6] and references therein)

$$i\hbar\frac{\partial\tilde{\phi}(\boldsymbol{p},t)}{\partial t}=\sqrt{\boldsymbol{p}^2c^2+m^2c^4}\,\tilde{\phi}(\boldsymbol{p},t), \tag{2.3}$$

and demand that the relativistic Wigner function has the basic properties of the nonrelativistic one referring to integration over the spatial and momentum variables such that

$$\int d^3\boldsymbol{p}W(\boldsymbol{x},\boldsymbol{p},t)=|\phi(\boldsymbol{x},t)|^2=\rho(\boldsymbol{x},t), \tag{2.4}$$

$$\int d^3\boldsymbol{x}W(\boldsymbol{x},\boldsymbol{p},t)=\frac{1}{\hbar^3}|\tilde{\phi}(\boldsymbol{p},t)|^2=\rho_p(\boldsymbol{p},t), \tag{2.5}$$

and satisfy the evolution law

$$W(\boldsymbol{x},\boldsymbol{p},t+\tau)=W\left(\boldsymbol{x}-\frac{c\boldsymbol{p}}{p_0}\tau,\boldsymbol{p},t\right), \tag{2.6}$$

where $p_0=E/c=\sqrt{\boldsymbol{p}^2+m^2c^2}$. It is easy to verify that the evolution law (2.6) is the global form of the local relation

$$\frac{\partial W(\boldsymbol{x},\boldsymbol{p},t)}{\partial t}+\frac{c\boldsymbol{p}}{p_0}\cdot\nabla W(\boldsymbol{x},\boldsymbol{p},t)=0 \tag{2.7}$$

generalizing the nonrelativistic equation that is valid in the case of the free evolution

$$\frac{\partial W(\boldsymbol{x},\boldsymbol{p},t)}{\partial t}+\frac{\boldsymbol{p}}{m}\cdot\nabla W(\boldsymbol{x},\boldsymbol{p},t)=0. \tag{2.8}$$

With these assumptions the following relativistic generalization of (2.1) was obtained in [2]:

$$W(\boldsymbol{x},\boldsymbol{p},t)=\frac{1}{(2\pi)^3}\frac{1}{\hbar^6}\int d^3\boldsymbol{p}_1d^3\boldsymbol{p}_2\tilde{\phi}^*(\boldsymbol{p}_1,t)\tilde{\phi}(\boldsymbol{p}_2,t)\delta(\boldsymbol{p}-(\boldsymbol{p}_1\oplus\boldsymbol{p}_2))e^{\frac{i(\boldsymbol{p}_2-\boldsymbol{p}_1)\cdot\boldsymbol{x}}{\hbar}}, \tag{2.9}$$