Annals of Physics 375 (2016) 16–35

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Three paths toward the quantum angle operator

Jean Pierre Gazeau [a,*], Franciszek Hugon Szafraniec [b]

[a] Laboratoire APC, Univ Paris Diderot, Sorbonne Paris Cité, 75205 Paris, France
[b] Instytut Matematyki, Uniwersytet Jagielloński, 30-348 Kraków, Poland

ARTICLE INFO

Article history:
Received 7 March 2016
Accepted 26 September 2016
Available online 30 September 2016

Keywords:
Quantum angle
Quantum phase
Weyl–Heisenberg
Integral quantization
Coherent states
Self-adjointness

ABSTRACT

We examine mathematical questions around angle (or phase) operator associated with a number operator through a short list of basic requirements. We implement three methods of construction of quantum angle. The first one is based on operator theory and parallels the definition of angle for the upper half-circle through its cosine and completed by a sign inversion. The two other methods are integral quantization generalizing in a certain sense the Berezin–Klauder approaches. One method pertains to Weyl–Heisenberg integral quantization of the plane viewed as the phase space of the motion on the line. It depends on a family of "weight" functions on the plane. The third method rests upon coherent state quantization of the cylinder viewed as the phase space of the motion on the circle. The construction of these coherent states depends on a family of probability distributions on the line.

1. Introduction

We revisit the delicate and longstanding question of angular or phase localization on a quantum level, a problem considered by many authors since the birth of quantum physics [1–10] and recently examined on more mathematically oriented bases by Busch and Lahti [11] and Galapon [12] and the recent [13] (see also [14]). Closely related to this question is the validity of commutation relations between phase ($\sim$ angle) operator and number operator ($\sim$ angular momentum) in terms of their respective domains. For a recent alternative to the phase-number commutation rule and the associated uncertainty relations, see [15] and references therein.

* Corresponding author.
E-mail addresses: gazeau@apc.univ-paris7.fr (J.P. Gazeau), franciszek.szafraniec@uj.edu.pl (F.H. Szafraniec).

http://dx.doi.org/10.1016/j.aop.2016.09.010
0003-4916/

In Section 2 we propose a short list of requirements which seem to be natural in defining a proper angle operator coupled with a number-like operator. In Section 3 we start from the formal canonical commutation rules between ladder operators on a separable Hilbert space $\mathcal{H}$ to infer commutation rules involving extended number operator N. Once corresponding domains are well defined we display an angle operator A which is conjugate to N "modulo" partial isometry. By this we mean that their commutator reads $[N, A] = \mathrm{i}\,\Sigma$ where Σ is a partial isometry in $\mathcal{H} \oplus \mathcal{H}$. In a certain sense, this approach parallels the definition of angle for the upper half-circle through its cosine and completed by a sign inversion. In Section 4, starting from the classical angle of polar coordinates of the plane, viewed for instance as the phase space for the motion on the line, we follow a quite different procedure which we call Weyl–Heisenberg integral quantization [16], based on positive operator valued measure solving the identity. The issue is a family of bounded covariant self-adjoint operators with continuous spectrum supported by $[0, 2\pi]$. In Section 5 we consider the angular position for the motion on the circle and build its quantum counterpart by using families of coherent states for the circle derived from probability distributions on the real line [14]. Section 6 gives a short summary of our results and an insight on the continuation of our exploration.

2. Requirements for angle operator

Let us be more precise about the questions we are going to consider in our paper.

• *Angle function a*. On a classical level, we mean by an *angle* (or *phase*) *function* the 2π-periodic function on the real line such that $a(\gamma) = \gamma$ for $\gamma \in [0, 2\pi)$. The function a has the following property:

$$a(\gamma + \theta) = a(\gamma) + \theta \mod 2\pi, \tag{1}$$

notice, here "mod 2π" applies both to the independent variables and the values of the functions.

• *Angle operator A*. The angle or phase operator A, acting on some separable Hilbert space $\mathcal{H}$, is a quantum version of the angle function a restricted to the interval $[0, 2\pi)$, which is obtained through a quantization procedure; uniqueness is not discussed here. This operator is required to have the following properties.

(i) A is bounded self-adjoint on $\mathcal{H}$.

(ii) Its spectral measure is supported on the interval $[0, 2\pi)$:

$$A = \int_{[0,2\pi]} \gamma\, E_a(\mathrm{d}\gamma).$$

(iii) With a strongly continuous group $\{U_\theta\}_{\theta\in\mathbb{R}}$ of unitary operators and a partial isometry Σ given we have

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(U_\theta A U_{-\theta})|_{\theta=0} = -\Sigma, \quad \theta \in \mathbb{R}, \tag{2}$$

and subsequently with K being the generator of the group $\{U_\theta\}_{\theta\in\mathbb{R}}$, that is $U_\theta = \mathrm{e}^{\mathrm{i}\theta K}$,

$$AK - KA = \mathrm{i}\,\Sigma. \tag{3}$$

An alternative version of (2) is

$$U_{-\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}(U_\theta A U_{-\theta})U_\theta = -\Sigma, \quad \theta \in \mathbb{R}. \tag{2bis}$$

The properties (2) and (3) can be read precisely as *covariance* of A with respect to the group $\{U_\theta\}_{\theta\in\mathbb{R}}$ and the partial isometry Σ. It parallels that for a, that is (1).

On the basis of these natural requirements, we explore in this paper different ways to construct angle operator(s) fulfilling some if not all of these properties.