Physics Letters A ••• (••••) •••–•••

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Canonical distillation of entanglement

Tamoghna Das, Asutosh Kumar, Amit Kumar Pal, Namrata Shukla, Aditi Sen(De), Ujjwal Sen

Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhunsi, Allahabad 211019, India

ARTICLE INFO

Article history:
Received 21 April 2017
Received in revised form 30 August 2017
Accepted 31 August 2017
Available online xxxx
Communicated by P.R. Holland

Keywords:
Entanglement distillation
Canonical energy constraint
Quantum spin models

ABSTRACT

Distilling highly entangled quantum states from weaker ones is a process that is crucial for efficient and long-distance quantum communication, and has implications for several other quantum information protocols. We introduce the notion of distillation under limited resources, and specifically focus on the energy constraint. The corresponding protocol, which we call the canonical distillation of entanglement, naturally leads to the set of canonically distillable states. We show that for non-interacting Hamiltonians, almost no states are canonically distillable, while the situation can be drastically different for interacting ones. Several paradigmatic Hamiltonians are considered for bipartite as well as multipartite canonical distillability. The results have potential applications for practical quantum communication devices.

1. Introduction

Over the last twenty five years or so, entangled quantum states shared between distant parties have been proved to be essential for several quantum protocols [1–3]. However, unavoidable destruction of quantum coherence due to noisy quantum channels diminishes the quality of the shared quantum state, thereby posing a challenge to the implementation of such protocols. Invention of distillation protocols [4,6–11,5] to purify highly entangled states from collection of states with relatively low entanglement has been proven crucial in order to overcome such difficulties in device independent quantum cryptography [12,13], quantum dense coding [14], and quantum teleportation [15] – the three pillars of quantum communication. Entanglement distillation is also indispensable in quantum repeater models [16], used to overcome the exponential scaling of the error probabilities with the length of the noisy quantum channel connecting distant parties sharing the quantum state. Existence of bound entangled (BE) states [7,8] – entangled states from which no pure entangled state can be obtained using local operations and classical communications (LOCC) – further highlights the importance of identifying distillable states. Entanglement distillation protocols have also been used in problems related to topological quantum memory [17]. Laboratory realization of single copy distillation has been performed and possible experimental proposal of multicopy distillation has been given [18–20].

There is a close correspondence between entanglement and energy [7,8,23–25,21,22]. Moreover, consideration of statistical ensembles of quantum states of a system in terms of various constraints on its energy and number of particles is crucial in several areas of physics, including in quantum communication. An important example is the classical capacity of a noiseless quantum channel [26,27,30,31,28,29,32–36] for transmitting classical information using quantum states. The classical capacity is quantified by the von Neumann entropy of the maximally mixed quantum state that can be sent through the noiseless quantum channel. The “Holevo bound” [26,27,30,31] dictates that at most n bits of classical information can be transmitted using n distinguishable qubits, thereby predicting an infinite capacity for infinite dimensional systems, such as the bosonic channels [28,29,32–36]. Since the energy required to achieve infinite capacity is also infinite, such non-physicality can be taken care of by calculating the capacity under appropriate energy constraints. Constraints on available energy can also be active in other quantum information protocols including infinite- as well as finite-dimensional systems and in particular may give rise to a novel understanding of the interplay between entanglement and energy. For example, to implement ideas like quantum repeaters for long-distance quantum state distribution, an energy-constrained protocol for the distillation of entanglement may be necessary. Evidently, in that case, the energy of the states involved in the distillation process must follow constraints according to the physical situation in hand, especially in the case of implementation of the protocol in the laboratory, where arbitrary amount of energy is not accessible. The logical choice of such constraints may include bounds on average

E-mail address: aditi@hri.res.in (A. Sen(De)).

http://dx.doi.org/10.1016/j.physleta.2017.08.065
0375-9601/© 2017 Published by Elsevier B.V.

energy, or maximum available energy of the quantum states. We believe that energy constraint will be important for consideration of entanglement-based quantum communication protocols, that require entanglement distillation as a part of the protocol.

In this paper, we consider the process of distillation of highly entangled quantum states of shared systems from weakly entangled ones within the realm of limited resources. Specifically, we propose that a distillation protocol have to be carried out under an energy constraint, and refer to it as "canonical" distillation. We prove that non-interacting Hamiltonians lead to situations where canonically distillable states form a set of measure zero. The situation, however, drastically changes with the inclusion of interaction terms. We consider several paradigmatic interacting Hamiltonians of spin-$\frac{1}{2}$ systems, viz. the transverse-field XY model [37–42], the longitudinal-field XY model, and the XXZ model in an applied field [43,44], and the concept of canonical distillation is probed in each case. The interrelation between canonical distillability and the temperature in thermal states is also investigated. The findings are generic in the sense that they hold also in higher dimensions and for higher number of parties. The energy constraint in these cases is introduced by respectively considering a bilinear-biquadratic Hamiltonian [45–47] of two spin-1 particles and a multisite transverse XY model.

The paper is organized as follows. In Sec. 2, we define the canonical distillability of bipartite as well as multipartite quantum states. Sec. 3 contains the results on application of the canonical distillation protocol in bipartite systems. The results are also demonstrated in the cases of well-known quantum spin models, where the canonical distillability of pure and mixed states with respect to these Hamiltonians are tested. In Sec. 4, we discuss the canonical distillability of multipartite states, focusing on three-qubit pure states belonging to the Greenberger–Horne–Zeilinger (GHZ) [48,49] and the W [49,50] classes. Sec. 5 contains the concluding remarks.

2. Distillation under canonical energy constraint

We begin by providing a formal definition of *canonical* distillation of entanglement for two-qubit systems in the asymptotic limit. Generalization to higher dimensions and higher number of parties are considered later. In "usual" entanglement distillation [4,6], one intends to produce the largest number, m, of copies of the maximally entangled Bell pair, $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$, starting from n ($m \leq n$) copies of an entangled two-qubit state, ρ, using only LOCC. Let us consider an LOCC on n copies of the state ρ that creates the state σ which is close to m copies of $|\psi^-\rangle$, or its local unitary equivalent, $|\tilde{\psi}^-\rangle = \mathcal{U}_4|\psi^-\rangle$, so that

$$\lim_{n\to\infty} \mathrm{tr}(\sigma\tilde{\sigma}^{\otimes m}) = 1, \quad (1)$$

where $\tilde{\sigma} = |\tilde{\psi}^-\rangle\langle\tilde{\psi}^-|$. Here, $\mathcal{U}_4 = U_2^1 \otimes U_2^2$, with U_2^1 and U_2^2 being unitary operators on the qubit Hilbert space. The distillable entanglement is given by $E_D = \max \lim_{n\to\infty} \frac{m}{n}$, where the maximum is over all LOCC protocols satisfying Eq. (1).

To introduce an appropriate energy constraint, suppose that the two-qubit quantum system in the state ρ is described by the Hamiltonian H. Here, by "system", we mean the quantum system containing the set of n resource states, in turn containing the set of m output states of the distillation protocol. We assume that the system is in contact with a heat bath such that the average energies of the input and output states of the distillation protocol are equal. This average energy conservation leads to the constraint $\mathrm{tr}(\tilde{H}_n\rho^{\otimes n}) = \mathrm{tr}(\tilde{H}_m\sigma) \approx \mathrm{tr}(\tilde{H}_m\tilde{\sigma}^{\otimes m})$, with $\tilde{H}_j = \sum_{i=1}^{j} I^{\otimes i-1} \otimes H \otimes I^{\otimes j-i}$, which implies

$$\mathrm{tr}(H\rho) = \frac{m}{n}\left(\mathrm{tr}(H\tilde{\sigma})\right). \quad (2)$$

Here we assume that n is sufficiently large, so that $\mathrm{tr}(\tilde{H}_m\sigma)$ can be approximated by $\mathrm{tr}(\tilde{H}_m\tilde{\sigma}^{\otimes m})$. It can be shown, by virtue of Eq. (1), that the approximation is an equality for $n \to \infty$.

Note that we are assuming an insignificant contribution in average energy from the $n - m$ bipartite systems that are traced out, and any additional ancillary systems that are used and then discarded out during the LOCC protocol for the canonical distillation. Such energy dissipation channels can be incorporated into the definition, but leads to further intractability in the analysis. On the other hand, this assumption can be justified by noticing that the remnants after the application of a usual distillation protocol for creating singlet from pure two-qubit non-maximally entangled states [5], $\alpha|00\rangle + \beta|11\rangle$, are of the form $|0\rangle_A^{\otimes n}|0\rangle_B^{\otimes n}$ and $|1\rangle_A^{\otimes n}|1\rangle_B^{\otimes n}$ with probabilities $|\alpha|^{2n}$ and $|\beta|^{2n}$, respectively, where A and B are the two parties. This contributes in average energy of the system by an amount δ_E, where $\delta_E = n[|\alpha|^{2n}\langle 0_A 0_B|H|0_A 0_B\rangle] + |\beta|^{2n}\langle 1_A 1_B|H|1_A 1_B\rangle]$. Since $0 \leq |\alpha|, |\beta| \leq 1$, for $|\alpha|, |\beta| \neq 0, 1$, $\delta_E \to 0$ as $n \to \infty$. We will discuss specific examples in the coming sections, where we consider several important and specific forms of the system Hamiltonian. In the limit $n \to \infty$, from Eq. (2), we have

$$\mathrm{tr}(H\rho) = \lim_{n\to\infty} \frac{m}{n}\mathrm{tr}(H\tilde{\sigma}). \quad (3)$$

The average energy constraint can also be replaced by a maximal available energy constraint, wherein we expect the broad qualitative features, of the case where the average energy is considered, to be retained. The canonically distillable entanglement, E_{CD}, is the maximum value of $\lim_{n\to\infty} \frac{m}{n}$ that satisfies Eq. (3) for some $\mathcal{U}_4$, and is consistent with Eq. (1). We call the states with a non-zero E_{CD} to be canonically distillable (CD). One must note that for the two-qubit systems,

$$0 \leq E_{CD} \leq E_D \leq 1. \quad (4)$$

We would like to emphasize here that the canonical energy constraint, in the present problem, is imposed on the ensemble of quantum states over which the LOCC protocol is applied. In our approach, we consider local operations and classical communication (LOCC) which can be represented by two Hamiltonians. (1) The first one is the "environment" Hamiltonian describing, in a real experiment, the laboratory setting to implement the LOCC protocol. (2) The second one is the interacting Hamiltonian which models the interaction between the system, which in our case is the initial quantum states to be distilled, and the environment in the same experiment. We do not claim that the energy exchanges due to these two Hamiltonians are not important. However, we only consider the Hamiltonian governing the interaction between the systems from which the distillation is to occur. We consider this as a first step towards considering a general canonical distillation process, where the Hamiltonians in (1) and (2) can also be considered. Examples where such first steps have been considered and have provided important information about the systems are as follows: (a) In bosonic channels, a physically relevant bosonic channel capacity is obtained by providing an energy constraint on the source states (see, Refs. [33,34,36]). However, ideally, one should put an energy constraint on the entire process of encoding, sending, and decoding of the channel states, which is mathematically more challenging. Despite the disregard of the energy exchanges due to the system-environment interacting Hamiltonian as well as the environment Hamiltonian, a physically reasonable channel capacity is obtained. In a similar fashion, instead of considering the novel constraints put over the average energy of the source states by the environment Hamiltonian and the Hamiltonian due to the interaction of the system and the environment, in the distillation process, we assume that the system has already equilibrated with its laboratory environment, so that the average energy constraint applied