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On planar compactons with an extended regularity

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ABSTRACT

Using a Lotka–Volterra type system on a hexagonal lattice we derive and study a novel, strongly nonlinear dispersive equation $u_t = \partial_x(u + \Delta u)^n$, $n > 1$, the n-Cubic equation, which supports the formation and propagation of planar compactons endowed with extended regularity at their perimeter. Compactons may be uni-modal or, if n is odd, multi-modal as well. Both evolution and interaction of compactons are presented and discussed.

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1. Introduction

The subject of this communication are planar genuinely nonlinear dispersive "n-Cubic" equations [1]

$$\frac{\partial u(\vec{x},t)}{\partial t} = \frac{\partial}{\partial x}\left(\left(\mathcal{L}u(\vec{x},t)\right)^n\right), \quad n > 1 \tag{1.1}$$

where

$$\mathcal{L} \doteq 1 + \Delta, \quad \Delta \doteq \partial_x^2 + \partial_y^2 \quad \text{and} \quad \vec{x} = (x, y),$$

which balance multi-dimensional dispersion with convection in x direction (e.g. direction of gravity). The one dimensional rendition of (1.1) has been derived and studied in [2] as a quasi-continuum approximation of a class of Lotka–Volterra lattices. A similar derivation for the planar case will be presented in the next section. A family of planar compactons were originally introduced and studied in [3]. The most notable feature of the n-Cubic equations to be studied is the extended smoothness of their compactons as compared with previously studied models with similarly nonlinear dispersion.

The plan of the paper is as follows: In section 2 which is the core of the paper we

- Derive Eq. (1.1) as a quasi-continuum rendition of a hexagonal lattice.
- Unfold the underlying Hamiltonian structure of Eq. (1.1) and the consequent conservation laws.
- Derive the traveling waves of Eq. (1.1) and confirm the extended regularity of the radially symmetric compactons. A noteworthy feature of Eq. (1.1) is the countable number of multi-modal compactons available for odd n's.

In section 3 we study numerically evolution, interaction and general dynamics of compactons and in section 4 we summarize our results. In the Appendix we introduce a modified Petviashvili algorithm to compute the traveling waves on a planar lattice.

Finally, since emergence of compactons and their regularity is the main subject of the paper, it behooves us to provide a brief overview of compactons bearing equations. Compactons have been introduced in [4] as traveling solitary wave solutions with a compact support of the nonlinear dispersive $1+1$ dimensional $K(m,n)$ equations

$$K(m,n): \quad v_t = (v^m)_x + (v^n)_{3x}, \quad v = v(x,t). \tag{1.2}$$

The $N+1$ dimensional $\mathcal{C}_N(m,a,b)$ equations [5],

$$\mathcal{C}_N(m,a,b): \ v_t = \left(v^m\right)_x + \left(v^a \Delta v^b\right)_x, \ \ v = v(x,t,x_2,x_3,\ldots,x_N) \tag{1.3}$$

extend the $K(m,n)$ equations into a broader class of third order nonlinear dispersive equations, both in the form of the nonlinearity and the spatial dimension. Let $s = x + \lambda t$ and $r = \sqrt{s^2 + \sum_{i=2}^N x_i^2}$. Then radially symmetric traveling waves $v_c(r)$ of (1.3) satisfy

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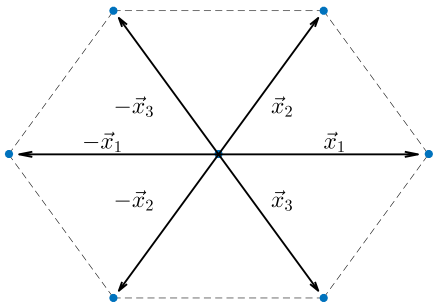

Fig. 1. Hexagonal lattice unit cell.

$$\lambda v_c = v_c^m + v_c^a \Delta_r v_c^b, \text{ where } \Delta_r = \partial_r^2 + \frac{N-1}{r}\partial_r. \tag{1.4}$$

Their degree of smoothness at the boundaries is then determined by the total degree of dispersive nonlinearity; $n = a + b$, [3,5,6]. At the edge of the support of a compacton one finds [5]

$$v_c(r) \sim r^{2/(n-1)} H(r), \tag{1.5}$$

where $H(s) = \begin{cases} 1, & s \ge 0 \\ 0, & s < 0 \end{cases}$ is the Heaviside function. Note that the degree of smoothness is independent of the spatial dimension N.

The summarized features will serve as a reference for comparison with the compactons to be presented in the following sections.

2. Properties of the planar *n*-cubic equations

2.1. Lattice quasi-continuum

In 1D, the n-Cubic equations emerge as a quasi-continuum approximation of a Lotka–Volterra type chain [2,7,8]

$$\begin{aligned} h\dot{U}_j &= \mathcal{A}(U_j, U_{j\pm1})\left(U_{j+1} - U_{j-1}\right), \\ \mathcal{A}(U_j, U_{j\pm1}) &\doteq a_0 U_j + a_1\left(U_{j+1} + U_{j-1}\right). \end{aligned} \tag{2.1}$$

Models like (2.1) emerge in biological, physical, or social interactions studies [9–13]. When motion takes place on a zero background the chain (2.1) has been shown to support discrete traveling waves, *discretons*, which decay at a doubly exponential rate (contrast it with the usual exponential decay of solitons and their discrete counterparts). In models with $a_1 = 0$, Eq. (2.1) is sometimes referred to as discrete KdV. The motion is then possible only on a non-trivial background.

Here we reconsider a dynamical system introduced in [14], which extends the Lotka–Volterra chain (2.1) into a planar lattice. The hexagonal lattice (see Fig. 1) is a natural discrete antecedent of nonlinear isotropic continuum models being, in fact, *the only* 2D lattice which in the $h \to 0$ limit yields a continuum with isotropic solitary patterns (see discussion in section III.A of [15]). In fact, the isotropy will be manifested in the rotationally invariant form of the equation governing the structure of solitary waves (2.16). We thus discuss a lattice with the basis vectors

$$\vec{x}_1 = h(1,0), \quad \vec{x}_2 = \frac{h}{2}(1,\sqrt{3}), \quad \vec{x}_3 = \frac{h}{2}(1,-\sqrt{3}). \tag{2.2}$$

A planar extension of (2.1) is then

$$h\dot{U}_o = \mathcal{A}\cdot\nabla_d U, \text{ where } \mathcal{A} \doteq \frac{1}{3}\left(U_o + \frac{1}{6}\sum_{i=1}^{3}(U_{\vec{x}_i} + U_{-\vec{x}_i})\right) \tag{2.3}$$

$$\text{and} \quad \nabla_d U \doteq 2(U_{\vec{x}_1} - U_{-\vec{x}_1}) + \sum_{i=2}^{3}(U_{\vec{x}_i} - U_{-\vec{x}_i}).$$

Given the complexity of the system (2.3), to gain insight into its dynamics we imitate it with a quasi continuum. Expanding Eq. (2.3) yields

$$\begin{aligned} U_t &= \left(2U + \frac{h^2}{4}\Delta U + \mathcal{O}(h^4)\right)\left(2U_x + \frac{h^2}{4}\Delta U_x + \mathcal{O}(h^4)\right) \\ &= 2\left(U^2\right)_x + \frac{h^2}{2}(U\Delta U)_x + \frac{h^4}{32}\left((\Delta U)^2\right)_x \\ &\quad + \frac{h^4}{32}\left(U\Delta^2 U\right)_x - \frac{h^4 U}{120}\left(5\partial_x^2\partial_y^2 + \partial_x^4\right)U_x + \mathcal{O}(h^6). \end{aligned} \tag{2.4}$$

In a conventional weakly nonlinear approach one sets $U = 1 + \epsilon u$, $\epsilon \ll 1$:

$$u_t = 4u_x + 4\epsilon u u_x + \frac{h^2}{2}\Delta u_x + \mathcal{O}(\epsilon h^2) \tag{2.5}$$

and balances nonlinearity with dispersion via $\epsilon = h^2/8$. Define $\tilde{x} = x + 4t$ and $\tilde{t} = h^2 t/2$ and restore the original variables to obtain, up to $\mathcal{O}(\epsilon)$;

$$u_t = u u_x + \Delta u_x \tag{2.6}$$

which is the Zakharov–Kuznetsov, ZK, equation [16], known to support planar solitary waves.

Since we address a genuinely nonlinear regime *on a zero background* we apply the expansion (2.4) as is. If, cf. [14], only $\mathcal{O}(h^2)$ terms are retained, then after the normalization

$$\tilde{x} = \frac{2\sqrt{2}}{h}x, \quad \tilde{y} = \frac{2\sqrt{2}}{h}y, \quad \tilde{t} = \frac{2\sqrt{2}}{h}t, \tag{2.7}$$

and $\tilde{x} \to x$, $\tilde{y} \to y$, $\tilde{t} \to t$, $U = v/2$, one has

$$v_t = (v^2)_x + 2(v\Delta v)_x \tag{2.8}$$

which, up to a normalization, is the $\mathcal{C}_2(2,1,1)$ equation, see section 1 (the coefficient 2 was retained for a later use). Eq. (2.8) has the explicit compacton solution

$$\begin{aligned} v(r,t) &= \lambda\left(1 - \frac{J_0(r/\sqrt{2})}{J_0(\xi_1)}\right) H(\sqrt{2}\xi_1 - r), \\ r &= \sqrt{(x+\lambda t)^2 + y^2}, \end{aligned} \tag{2.9}$$

where $\xi_1 > 0$ is the location of the first zero of J_1. At the edge of the compacton v and v_r vanish but the second order derivatives of v have a jump discontinuity.

The next step is a bit unorthodox; since $x \to x/h$, $y \to y/h$ eliminates h *to all orders* from expansion (2.4), the expansion is not asymptotic. Thus there is no particular significance to terminate the expansion at a particular order of h. With an equal justification one may reorder the expansion using another organizing rule. Our choice is to retain all terms up to third order spatial derivatives, the second row in (2.4), thus keeping the same degree of complexity in derivatives. This amounts to adding the $\frac{h^4}{32}\left((\Delta u)^2\right)_x$ term which, surprisingly enough, enables to cast the problem into the simple form

$$U_t = 2\left(\left(U + \frac{h^2}{8}\Delta U\right)^2\right)_x.$$

Using normalization (2.7) and restoring the original variables with $U = w/2$ yields the $n = 2$ variant of the n-Cubic equation

$$w_t = \left((w + \Delta w)^2\right)_x.$$

An n-Cubic extension may be similarly derived via a lattice of the form

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