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11 Event colution of a classical short range spin model with a phase $\frac{11}{12}$ Exact solution of a classical short-range spin model with a phase $\frac{77}{78}$ ¹³ transition in one dimension: The Potts model with invisible states **1988** 14

15 81 Petro Sarkanych ^a*,*b*,*1, Yurij Holovatch ^a*,*1, Ralph Kenna ^b*,*¹ 16 82

17 83 ^a *Institute for Condensed Matter Physics, National Academy of Sciences of Ukraine, Lviv, Ukraine* 18 84 ^b *Applied Mathematics Research Centre, Coventry University, Coventry, England, United Kingdom*

20 and the contract of the con zi ARTICLE INFO ABSTRACT 87

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entropy, but not to the interaction energy. We determine the partition function, using the transfer-matrix 25 Accepted 29 August 2017
nethod, in the general case of two ordering fields: h_1 acting on a visible state and h_2 on an invisible 26 Available online xxxx
₂₇ Communicated by L. Ghivelder **1892 State.** We analyse its zeros in the complex-temperature plane in the case that $h_1 = 0$. When Im $h_2 = 0$ $\frac{27}{27}$ communicated by E. Ginverser 28 *Keywords:* $\frac{1}{2}$ **11** $\frac{1}{2}$ **12** $\frac{1}{2}$ **12** $\frac{1}{2}$ $\frac{1}{2$ 29 Phase transitions the section of zeros intersects the positive part of the real temperature axis, which signals the existence of a phase the exi 30 96 transition at non-zero temperature. the *q* states of the ordinary Potts model, this possesses *r* additional states which contribute to the

 $_{31}$ Potts model and the set of the set of

35 **1. Introduction**

45 1 In the 1920s Wilhelm Lenz and Ernst Ising suggested and in- Beucally lavourable for domain walls to be inserted. This means 111 46 vestigated the first microscopic model of ferromagnetism [\[7,8\].](#page--1-0) The system cannot become ordered and there is no phase transi-47 Theirs involved a one-dimensional lattice occupied by classical and the such the systems. 48 spins which can only be in one of two states: up or down. In the wall nove proved the absence of a phase danshion in one-49 short-range version, only nearest neighbouring spins are allowed and differential mud-like systems of particles with non-valushing in-50 to interact. Ising showed that this model has no phase transition compressionly radius and a mille range or loces [12]. This was fire 51 at any physically accessible (i.e., non-zero) temperature T [\[8\].](#page--1-0) This extended by Kuene [15] to facture models. These results are based 117 52 served as the first and archetypal example of the absence of a $\frac{61}{118}$ for the reflori-riobelities theorem [14]. However, and as empha-53 119 finite-*T* phase transition in 1D classical systems with short-range 54 interactions [\[9\].](#page--1-0) Later, in their famous book [\[10\],](#page--1-0) Landau and Lif- The existence of thermodynamic phase transitions in general 1D - 12c 55 shitz gave a heuristic argument that for classical systems there is systems with short-range interactions [15]. Indeed Cuesta and 121 56 122 no phase transition at non-zero temperature in 1D. The approach 57 is based on separately evaluating contributions to the free energy discussed now the no-go theorems might be circumvented [16]. In 123

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₃₆ **1. Introduction S**. For the Ising model there are two ground states in which all ₁₀₂ $\frac{37}{103}$ spins are either up or down. At zero temperature the system is in 38 10 Few models in statistical physics can be solved exactly and 10 one of these states. At finite temperatures, domain walls separate 104 39 many of those that can are in a single dimension [\[1\].](#page--1-0) Knowing the equals of up- and down-spins. Each domain wan costs energy $_{105}$ 40 behaviour of a system in one dimension (1D) can help one to un- then there assess the interaction energy E). But in the TD case $\frac{106}{100}$ 41 derstand and predict its behaviour in higher dimensions too. Also this is outweighed by the entropic contribution coming from the 107 42 some chemical compounds are effectively described by quasi-1D and the mays of placing domain wails on the chain. See, e.g., are 43 models $[2-6]$. For these reasons, 1D models are interesting from kel. [11] where it is explicitly shown that adding domain walls 105 44 both theoretical and experimental points of view.
Teques the frequency of the reference, at any temperature it is energy from the reference of the reference of one of these states. At finite temperatures, domain walls separate regions of up- and down-spins. Each domain wall "costs" energy (i.e., it increases the interaction energy *E*). But in the 1D case this is outweighed by the entropic contribution coming from the number of ways of placing domain walls on the chain. See, e.g., Ref. [\[11\]](#page--1-0) where it is explicitly shown that adding domain walls reduces the free energy. Therefore, at any temperature it is energetically favourable for domain walls to be inserted. This means the system cannot become ordered and there is no phase transition in such 1D systems.

58 124 *F* = *E* − *T S* coming from the interaction energy *E* and the entropy 59 125 in field are mapped onto corresponding zero-field 2D models with 60 126 nearest neighbours interactions. Additionally, quantum 1D models 61 $\overline{F_{t}}$ $\overline{F_{t}}$ $\overline{F_{t}}$ and address: sarkanyo@uni coventry ac uk (P. Sarkanyob) 62 $1 \t L⁴$ Collaboration & Doctoral College for the Statistical Physics of Complex Sys-
Sition at non-zero temperatures because they are they are related to 2D and the Subvan Hove proved the absence of a phase transition in onedimensional fluid-like systems of particles with non-vanishing incompressibility radius and a finite range of forces [\[12\].](#page--1-0) This was extended by Ruelle [\[13\]](#page--1-0) to lattice models. These results are based on the Perron–Frobenius theorem [\[14\].](#page--1-0) However, and as emphasised by Cuesta and Sánchez, none of these theorems preclude the existence of thermodynamic phase transitions in general 1D systems with short-range interactions [\[15\].](#page--1-0) Indeed Cuesta and Sánchez gave examples of such models and Theodorakopoulos also discussed how the no-go theorems might be circumvented [\[16\].](#page--1-0) In Refs. [\[17,18\]](#page--1-0) the 1D multispin-interaction Ising and Potts models sition at non-zero temperatures because they are related to 2D classical systems [\[19\].](#page--1-0)

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E-mail address: sarkanyp@uni.coventry.ac.uk (P. Sarkanych).

⁶³ 129 tems, Leipzig–Lorraine–Lviv–Coventry, Europe.

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9 75 where *Z(*3*)* symmetry is broken. It differs from the ordinary Potts ¹⁰ model [\[22,23\]](#page--1-0) by adding non-interacting states; if a spin is in one The rest of the paper is organised as follows: in Section 2 we 76 sponding Hamiltonian was written in the form

$$
H = -\sum_{\langle i,j\rangle} \delta_{S_i,S_j} \sum_{\alpha=1}^q \delta_{S_i,\alpha} \delta_{S_j,\alpha},
$$
\n(1)\nconclusions are given in Section 4.\n
$$
H = -\sum_{\langle i,j\rangle} \delta_{S_i,S_j} \sum_{\alpha=1}^q \delta_{S_i,\alpha} \delta_{S_j,\alpha},
$$
\n(2)\n
$$
B = -\sum_{\langle i,j\rangle} \delta_{S_i,S_j} \sum_{\alpha=1}^q \delta_{S_i,\alpha} \delta_{S_j,\alpha},
$$
\n(3)\n
$$
B = -\sum_{\langle i,j\rangle} \delta_{S_i,S_j} \sum_{\alpha=1}^q \delta_{S_i,\alpha} \delta_{S_j,\alpha},
$$
\n(4)\n
$$
B = -\sum_{\langle i,j\rangle} \delta_{S_i,S_j} \sum_{\alpha=1}^q \delta_{S_i,\alpha} \delta_{S_j,\alpha},
$$
\n(5)

¹⁷ where *q* and *r* are the number of visible and invisible states of the $\frac{83}{24}$ 18 84 Let us consider the *(q,r)*-state Potts model on a chain con-Potts variable

$$
s_i = 1, ..., q, q + 1, ..., q + r.
$$
 (2)

²² The first sum in (1) is taken over all distinct pairs of interacting ary conditions, the Hamiltonian of such a system may be written ²³ particles, and the second sum requires both of the interacting spins $\frac{1}{25}$ as 24 to be in the same visible state. From now and on we will use the 90 ²⁵ notation (q, r) -state Potts model for model with *q* visible and *r* 26 invisible states $H(a_r) = -\sum \delta_{s_r} \delta_{s_r} \delta_{\alpha} s_{s_{r+1}} - h_1 \sum \delta_{s_r} \delta_{r-1} - h_2 \sum \delta_{s_{r-1}} \delta_{r-1}$, (4) 92 invisible states.

27 In the ordinary Potts model [\[22\],](#page--1-0) the parameter $q \ge 2$ has been $i \alpha = 1$ $i \alpha = 1$ ²⁸ introduced as an integer denoting the number of (visible) states where the cum over i is taken over all sites of the chain 29 that a site can be in. However, it has been extended to other val-
 $\frac{W_{\text{LO}}}{W_{\text{LO}}}\frac{W_{\text{LO}}}{W_{\text{LO}}}\frac{W_{\text{LO}}}{W_{\text{LO}}}\frac{W_{\text{LO}}}{W_{\text{LO}}}\frac{W_{\text{LO}}}{W_{\text{LO}}}\frac{W_{\text{LO}}}{W_{\text{LO}}}\frac{W_{\text{LO}}}{W_{\text{LO}}}\frac{W_{\text{LO}}}{W_{\text{LO}}}\frac{W_{\$ ³⁰ ues too in order to describe bond percolation $(q = 1)$, dilute spin
³⁰ the exact solution of the model (4). The Hamiltonian (4) can be $\frac{31}{9}$ glasses ($q = 1/2$), and gelation ($q = 0$) [\[23–25\].](#page--1-0) The parameter *q* expressed as a sum of terms representing one bond each, so that $\frac{32}{12}$ has been extended to complex values as well [\[26–29\].](#page--1-0) In a sim-
 $\frac{22}{12}$ has been extended to complex values as well [26–29]. In a sim-
 $\frac{22}{12}$ has been extended to complex values as well [26–29]. In a s ³³ ilar way, although an initial interpretation of the parameter *r* is $H(q,r) = \sum_i H_i$ where ³⁴ the number of invisible states, we extend it here to non-integer q and the number of invisible states, we extend it here to non-integer and the states of the ³⁵ and even negative values. As we will show below, the latter cor- $H_i = -\sum \delta_{\alpha} \alpha_{\alpha} \delta_{\alpha} \alpha_{\alpha} - h_1 \delta_{\alpha} \alpha_{\alpha} + h_2 \delta_{\alpha} \alpha_{\alpha} \alpha_{\alpha}$ (5) ¹⁰¹ ³⁶ responds to removing entropy from the system and will be key to $\frac{2}{\alpha-1}$, $\frac{1}{\alpha-2}$, $\frac{1}{\alpha}$, $\frac{1}{\alpha-1}$, $\frac{1}{\alpha-2}$, $\frac{1}{\alpha-1}$, $\frac{1}{\alpha-2}$, $\frac{1}{\alpha+1}$, $\frac{1}{\alpha-2}$, $\frac{1}{\alpha-1}$ 37 inducing a phase transition. 103

³⁸ Adding invisible states does not change the spectrum of the Then the partition function can be transformed to the 39 105 model, it only changes the degeneracy of energy levels (the num-⁴⁰ ber of configurations with a given energy). Even though invisible $Z = \sum \exp[-\beta H_{(q,r)}] = \sum \left[\exp(-\beta H_i) \right]$, (6) ¹⁰⁶ 41 states do not change the interaction energy, since they change the $\frac{107}{s}$ ⁴² entropy they affect the free energy. As a consequences, for exam-⁴³ ple, an increase in the number *r* of invisible states may cause a where $β = 1/kT$ and k is the Boltzmann constant. Now it is easy 109 ⁴⁴ bhase transition to change from second to first order [20.30]. For to define the $(q+r) \times (q+r)$ square transfer matrix with elements 110 45 111 example the *(*2*,* 30*)*-state model on a square lattice undergoes the ⁴⁶ first order transition, while the ordinary (2,0)-state Potts model $T_{ii} = \exp \left[\beta \left(\frac{\delta_{c}}{\delta_{c}} \sum_{n=1}^{\infty} \frac{\delta_{c}}{n} \frac{1}{n} \right] + b_1 \frac{\delta_{c}}{n} \frac{1}{n} \right]$ (7) ¹¹² 47 (i.e. the Ising model) is an iconic example of a continuous transi-
 $\left\{ \frac{\mu_1-\mu_2}{\mu_1-\mu_2} \right\}$ (i.e. the Ising model) is an iconic example of a continuous transi- $\alpha = 1$ $\alpha = 1$ $\alpha = 1$ $\alpha = 1$ $\alpha = 1$ ple, an increase in the number *r* of invisible states may cause a phase transition to change from second to first order [\[20,30\].](#page--1-0) For tion.

⁴⁹ The Potts model with invisible states describes a number of Let us denote **115** the post of the states of the states of the states and the states of the states of the states and the states of the states of the states 50 116 models of physical interest. In particular, the *(*1*,r)*-state model ⁵¹ can be mapped to the Ising model in a temperature-dependent $t = e^{-\beta}$, $z_1 = e^{\beta h_1}$, $z_2 = e^{\beta h_2}$. (8) ¹¹⁷ 52 118 field [\[31\].](#page--1-0) The 1D *(*1*,r)* case with nearest-neighbour interactions is 53 equivalent to the Zimm-Bregg model for the helix-coil transition With this notation, positive values of temperature T correspond 119 $\frac{54}{12}$ $\frac{120}{120}$ The multi-spin extension of this model possesses a re-entrant to t ranging from zero to one, and the elements of the transfer 120 $\frac{1}{25}$ the matrix can be written in the compact form: $T_{11} = z_1/t$; $T_{ii} = t^{-1}$ 121
phase transition and is in good agreement with experimental ob-
matrix can be written in the compact form: $T_{11} = z_1/t$; $T_{ii} = t^{-1}$ 56 servations for polymer transitions [32,33]. The $(2, r)$ -state Potts for $1 < i \leq q$; $T_{i1} = z_1$; $T_{(q+1)i} = z_2$; and all remaining elements 122 57 model without external fields is equivalent to the Blume–Emery–
 123 equivalent to the Zimm–Bregg model for the helix-coil transition [\[31\].](#page--1-0) The multi-spin extension of this model possesses a re-entrant phase transition and is in good agreement with experimental observations for polymer transitions [\[32,33\].](#page--1-0) The *(*2*,r)*-state Potts Grifiths (BEG) model [\[20,34,35\]](#page--1-0) with a temperature dependent external field. Furthermore, the general *q* and *r* case can be interpreted as a diluted Potts model [\[20,30\]](#page--1-0)

$$
H' = -\sum_{\langle i,j\rangle} \delta_{\sigma_i,\sigma_j} (1 - \delta_{\sigma_i,0}) - T \ln r \sum_i \delta_{\sigma_i,0},\tag{3}
$$

¹ In the light of these enduring discussions, it is interesting to The this paper we perform an analysis of the partition function ⁶⁷ ² investigate further circumstances in which classical systems might zeros to obtain the exact solution for the 1D Potts model with in- 68 ³ exhibit a phase transition in 1D. In particular, we are interested in visible states. In achieving this, we add another exact result to the 69 ⁴ a mechanism in which entropy production can be suppressed in existing collection of exactly solved models in statistical mechan- ⁷⁰ ⁵ order to sidestep the conditions of the arguments and theorems ics. As we will show below, although one-dimensional, the model ⁷¹ 6 discussed above. To do so we consider the Potts model with invis- manifests a transition at non-zero temperature provided some un- 72 ⁷ ible states which was introduced a few years ago [\[20,21\]](#page--1-0) in order usual conditions are assumed. We will discuss regimes in which 73 8 to explain some questions about the order of a phase transition such behaviour is observed and a possible connection with quan- 74 In this paper we perform an analysis of the partition function zeros to obtain the exact solution for the 1D Potts model with invisible states. In achieving this, we add another exact result to the existing collection of exactly solved models in statistical mechanics. As we will show below, although one-dimensional, the model manifests a transition at non-zero temperature provided some unusual conditions are assumed. We will discuss regimes in which such behaviour is observed and a possible connection with quantum systems.

¹¹ such state, it is "invisible" to its neighbours. Originally, the corre-
¹¹ such stang oution using the transfer-matrix method, then ⁷⁷ ¹² sponding Hamiltonian was written in the form $\frac{1}{2}$ in Section [3](#page--1-0) the existence of a phase transition at non-zero tem-13 79 perature is demonstrated using partition function zeros, and finally ¹⁴ \overline{a} \overline{b} \overline{c} \overline{d} \overline{c} conclusions are given in Section [4.](#page--1-0) The rest of the paper is organised as follows: in Section 2 we present the exact solution using the transfer-matrix method, then

19 85 sisting of *N* spins with only nearest-neighbour interactions subject $s_i = 1, \ldots, q, q + 1, \ldots, q + r$.

(2) to two separate magnetic fields h_1 and h_2 acting on the first visible $\frac{21}{21}$ and the first invisible states respectively. Imposing periodic boundary conditions, the Hamiltonian of such a system may be written as

$$
H_{(q,r)} = -\sum_{i} \sum_{\alpha=1}^{q} \delta_{s_i,\alpha} \delta_{\alpha,s_{i+1}} - h_1 \sum_{i} \delta_{s_i,1} - h_2 \sum_{i} \delta_{s_i,q+1}, \quad (4)
$$

where the sum over *i* is taken over all sites of the chain.

We will use the transfer matrix formalism [36-38] to obtain the exact solution of the model (4) . The Hamiltonian (4) can be expressed as a sum of terms representing one bond each, so that $H_{(q,r)} = \sum_i H_i$ where

$$
H_{i} = -\sum_{\alpha=1}^{q} \delta_{s_{i},\alpha} \delta_{\alpha,s_{i+1}} - h_{1} \delta_{s_{i},1} - h_{2} \delta_{s_{i},q+1}.
$$
 (5)

Then the partition function can be transformed to

$$
Z = \sum_{s} \exp\left[-\beta H_{(q,r)}\right] = \sum_{s} \prod_{i} \exp\left(-\beta H_{i}\right),\tag{6}
$$

$$
T_{ij} = \exp\left[\beta \left(\delta_{s_i,s_j} \sum_{\alpha=1}^q \delta_{s_i,\alpha} + h_1 \delta_{s_i,1} + h_2 \delta_{s_i,q+1}\right)\right].
$$
 (7)

Let us denote

$$
t = e^{-\beta}, \ z_1 = e^{\beta h_1}, \ z_2 = e^{\beta h_2}.
$$
 (8)

equal to 1.

⁵⁸ Crifiths (REC) model [20.34.35] with a temperature dependent Based on the transfer-matrix symmetry it is easy to show that 124 59 external field. Furthermore, the general q and r case can be in-
it has five different eigenvalues. The eigenvalue $\lambda_0 = 0$ is $(r - 1)$ 125 ⁶⁰ terpreted as a diluted Potts model [20,30] **track interventive state of the matrix are** the matrix are the matri 61 **127 1** $H' = -\sum \delta_{\sigma_i,\sigma_i}(1-\delta_{\sigma_i,0}) - T \ln r \sum \delta_{\sigma_i,0},$ (3) equal to *t*^{−1}, choosing $\lambda = t^{-1} - 1$ one can find *q* − 2 linear inde-
⁶² 63 $\frac{1}{2}$ $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$ pendent eigenvectors. This reduces the problem to the determina-64 130 method in the more eigenvalues. They can be found using invariant the state of three more eigenvalues. They can be found using invariant the state of the state of three more eigenvalues. They can be found using inva 65 where $\sigma_i = 0, 1, \ldots, q$ is a new spin variable and all invisible states permutations. The above considerations lead to the equation for 131 66 are gathered into one the zeroth state $\sigma_i = 0$. The three remaining eigenvalues: permutations. The above considerations lead to the equation for the three remaining eigenvalues:

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