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On the earliest jump unravelling of the spatial decoherence master equation

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ABSTRACT

Solution of free particle quantum master equation with spatial decoherence can be unravelled into stochastic quantum trajectories in many ways. The first example (Diósi, 1985) proposed a piecewise deterministic jump process for the wave function. While alternative unravellings, diffusive ones in particular, proved to be tractable analytically, the jump process 1985, also called orthojump, allows for few analytic results, it needs numeric methods as well. Here we prove that, similarly to diffusive unravellings, it is localizing the quantum state.

1. Introduction

A single Schrödinger particle becomes a simple open quantum system if the particle is interacting with a thermal reservoir. Its dynamics is given by a master equation which can take the following simple form valid typically at high temperatures:

$$\frac{d\hat{\rho}}{dt}=-\frac{i}{\hbar}[\hat{H},\hat{\rho}]-\frac{D}{\hbar^2}[\hat{x},[\hat{x},\hat{\rho}]], \tag{1}$$

where $\hat{H}=(\hat{p}^2/2m)$ is the particle's Hamiltonian, $\hat{x},\hat{p}$ are its coordinate and momentum resp., and D is the diffusion constant. Joos and Zeh suggested this equation as the simplest model of spatial decoherence [1] while at the time similar single particle master equations were known from various fields, cf., e.g., [2,3]. The Wigner function of $\hat{\rho}$ satisfies

$$\frac{d\rho(x,p)}{dt}=-\frac{p}{m}\partial_x\rho(x,p)+D\partial_p^2\rho(x,p), \tag{2}$$

which coincides with the classical Fokker–Planck–Kramers equation [4] in the high-temperature (diffusion dominated, frictionless) limit. This elucidates the importance of the master equation (1) as the quantized version of momentum diffusion. Accordingly, D is the coefficient of spatial decoherence as well as of momentum diffusion: the two effects are alternative interpretations of the non-Hamiltonian mechanism in the master equation. It is well-known that the classical diffusion (2) can be equivalently described by random trajectories (x_t,p_t) in phase space. The same concept applies to the master equation (1) as well. The stochastic *quantum trajectories* are featured by state vectors Ψ_t evolving by a stochastic process such that the stochastic mean

$$\mathbf{M}\Psi_t\Psi_t^\dagger=\hat{\rho}_t \tag{3}$$

satisfies the master equation (1). Then the quantum trajectories Ψ_t are said to *unravel* the master equation.

The unravelling is never unique, one can choose diffusive unravellings, jump unravellings, or even their combinations. The earliest unravelling was the orthojump process [5]. It turned out subsequently that any master equation possesses a standard jump and a standard diffusive unravelling [6]. All possible diffusive unravellings can be parametrized uniquely [7,8], each of them corresponds to a given structure of time-continuous monitoring the system in question [8]. Similar classification is still missing for jump unravellings.

While quantum trajectories became instrumental soon for quantum optics [9–11], their invention happened earlier in studies of foundations. In the nineteen-eighties, diffusive quantum trajectories were invented by Gisin to model quantum state collapse in a discrete system [12]. One of the present authors constructed jump [5] and diffusive [13] unravellings of the master equation (1) for his gravity-related spontaneous state collapse theories [14]

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and [15], respectively. (On three decades of various spontaneous collapse theories, all based on unravellings, see the recent review by Bassi et al. [16].)

Analytic proof was found for the wave function localization in diffusive quantum trajectories [17]. The wave function is approaching a steady localized shape for long times, as we recapitulate it below. Localization in the specific jump unravellings [5] has, however, never been studied. The problem is more complicated than the diffusive case because jumps will never allow for a steady shape. An analytic proof of localization has not yet been found, we shall rely on numeric (Monte-Carlo) simulations. Jump quantum trajectories of spatial decoherence were carefully studied by Gisin and Rigo [18], and in a sequence of works by Hornberger and co-workers [19–21] for modifications of the master equation (1) which included friction. Due to friction, quantum trajectories did reach a localized steady shape, calculable analytically. The effect and proof was bound to the presence of friction. Localization in the frictionless case (1) has remained to be studied in the present work.

We are going to study localization of quantum trajectories in both position and momentum. Consider the unitary transformation of a state Ψ to its centre-of-mass frame:

$$\widetilde{\Psi} = \exp\left(i\langle\hat{x}\rangle\hat{p} - i\langle\hat{p}\rangle\hat{x}\right)\Psi, \tag{4}$$

where the centre-of-mass state satisfies $\langle\widetilde{\Psi}|\hat{x}|\widetilde{\Psi}\rangle = 0$ and $\langle\widetilde{\Psi}|\hat{p}|\widetilde{\Psi}\rangle = 0$ by construction. Now we can define the centre-of-mass density matrix as follows:

$$\mathbf{M}\widetilde{\Psi}_t\widetilde{\Psi}_t^\dagger = \hat{\widetilde{\rho}}_t. \tag{5}$$

This matrix is non-negative and of unit trace, like common density matrices. Its evolution, however, is non-linear, completely different from the master equation (1) of the common density matrix $\hat{\rho}_t$. Since $\hat{\widetilde{\rho}}_t$ is unravelling specific, we can use it to characterize the unravelling specific average localization of quantum trajectories Ψ_t around their individual centre-of-mass $\langle\hat{x}\rangle_t, \langle\hat{p}\rangle_t$. We can define centre-of-mass (squared) spreads by $(\Delta\widetilde{x})^2 = \mathrm{Tr}(\hat{x}^2\hat{\widetilde{\rho}})$ and by $(\Delta\widetilde{p})^2 = \mathrm{Tr}(\hat{p}^2\hat{\widetilde{\rho}})$.

2. Diffusive unravelling

Following [13,17], consider the stochastic Schrödinger equation [17]:

$$d\Psi = -\frac{i}{\hbar}\hat{H}\Psi dt - \frac{D}{\hbar^2}(\hat{x}-\langle\hat{x}\rangle)^2\Psi + \frac{\sqrt{2D}}{\hbar}(\hat{x}-\langle\hat{x}\rangle)\Psi dW, \tag{6}$$

where dW is the Ito-differential of the Wiener stochastic process, satisfying $\mathbf{M}dW = 0$, $(dW)^2 = dt$. The solutions satisfy the condition (3) of unravelling. For long time, the centre-of-mass solutions converge to the following complex Gaussian wave packet:

$$\widetilde{\Psi}_\infty(x) = \frac{1}{(2\pi\sigma_\infty^2)^{1/4}}\exp\left(-(1-i)\frac{x^2}{4\sigma_\infty^2}\right) \tag{7}$$

of squared width

$$\sigma_\infty^2 = \sqrt{\frac{\hbar^3}{2Dm}}. \tag{8}$$

According to (7), the centre-of-mass density matrix (5) turns out to converge to a pure state:

$$\hat{\widetilde{\rho}}_\infty = \widetilde{\Psi}_\infty\widetilde{\Psi}_\infty^\dagger. \tag{9}$$

The coordinate and momentum spreads are given by

$$(\Delta\widetilde{x})^2 = \sigma_\infty^2, \quad (\Delta\widetilde{p})^2 = \frac{\hbar^2}{2\sigma_\infty^2}. \tag{10}$$

Asymptotic localization is thus the analytically calculable feature of the diffusive quantum trajectories of the simple spatial decoherence master equation (1). The centre-of-mass of $\hat{\widetilde{\rho}}_\infty$ is performing the following diffusive motion:

$$d\langle\hat{x}\rangle = \frac{1}{m}\langle\hat{p}\rangle dt + \sqrt{\frac{2\hbar}{m}}dW, \qquad d\langle\hat{p}\rangle = \sqrt{2D}dW. \tag{11}$$

Observe that the diffusion of the momentum is the classical one. On the contrary, the diffusion of the coordinate cannot happen classically, it is purely quantum.

It may be interesting to see how simple is to recover the common density matrix $\hat{\rho}_t$ in the specific case when we have $\Psi_0 = \widetilde{\Psi}_\infty$ initially. Only we have to solve the stochastic equations (11) with the initial laboratory values $\langle\hat{x}\rangle_0 = \langle\hat{p}\rangle_0 = 0$, and apply (12) to construct Ψ_t is the laboratory:

$$\Psi_t = \exp\left(-i\langle\hat{x}\rangle_t\hat{p} + i\langle\hat{p}\rangle_t\hat{x}\right)\widetilde{\Psi}_\infty. \tag{12}$$

Then we recover the common density matrix via (3).

3. Orthojump unravelling

For the sake of comparison with the diffusive unravelling, let us cast the jump unravelling of [5] into the form of a stochastic Schrödinger equation:

$$d\Psi = -\frac{i}{\hbar}\hat{H}\Psi dt - \frac{D}{\hbar^2}[(\hat{x}-\langle\hat{x}\rangle)^2 - \sigma^2]\Psi dt + \left(\frac{x-\langle\hat{x}\rangle}{\sigma} - 1\right)\Psi dN, \tag{13}$$

where $\sigma^2 = \langle(\hat{x}-\langle\hat{x}\rangle)^2\rangle$. dN stands for the Ito-differential of a Poisson process, satisfying $\mathbf{M}dN = 2D\sigma^2 dt$, $(dN)^2 = dN$. This equation corresponds to a piece-wise deterministic evolution of Ψ_t, interrupted by jumps at random times. In elementary terms, the mechanism is the following. Consider the deterministic non-linear Schrödinger equation

$$\frac{d\Phi}{dt} = -\frac{i}{\hbar}\hat{H}\Phi - \frac{D}{\hbar^2}[(\hat{x}-\langle\hat{x}\rangle)^2]\Phi. \tag{14}$$

[Note that this equation coincides with the deterministic part of the diffusive stochastic Schrödinger equation (6) and they share $\widetilde{\Psi}_\infty$ (7) as (normalized) steady-shape centre-of-mass solution.] Solve this non-linear Schrödinger equation for the initial condition $\Phi_0 = \Psi_0$ and define the physical quantum state by $\Psi_t = \Phi_t/\|\Psi_t\|$. Note that the norm of Φ is strictly decreasing:

$$\frac{d\|\Phi\|^2}{dt} = -\frac{2D}{\hbar^2}\sigma^2. \tag{15}$$

The probability of jump-free deterministic evolution is decreasing exactly with the norm $\|\Phi\|^2$, i.e., the probability rate of jump is $(2D/\hbar^2)\sigma^2$. When a jump occurs, the smooth deterministic evolution of $\Psi/\|\Psi\|$ is interrupted by the sudden change

$$\Phi \longrightarrow (\hat{x}-\langle\hat{x}\rangle)\Phi, \tag{16}$$

rendering the new state orthogonal to what it was before the jump (cf. also [11]). After the jump, the deterministic evolution (14) restarts and continues until the next jump, etc.

4. Numeric tests of orthojumps

We have performed MC simulations of 15000 orthojump quantum trajectories. With suitable choice of physical units, we can always take trivial parameters $\hbar = m = D = 1$ and that is what we did. Discretized position coordinate $x \in (-5, +5)$ into 256 bins, the