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Logically reversible measurements: Construction and application

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ABSTRACT

We show that for any von Neumann measurement, we can construct a logically reversible measurement such that Shannon entropies and quantum discords induced by the two measurements have compact connections. In particular, we prove that quantum discord for the logically reversible measurement is never less than that for the von Neumann measurement.

1. Introduction

Measurement, as envisaged, plays an inevitable role in quantum mechanics, and lies at the heart of "interpretational problem" of quantum mechanics. Nonetheless, different views of measurement almost universally agree on the measurement outcomes. A quantum measurement is described in terms of a complete set of positive operators for the system to be measured. A few examples of quantum measurement are von Neumann measurement [1] which consists of orthogonal projectors, positive-operator-valued measure (POVM) [2], unitarily reversible measurement [3,4], etc. The most general type of measurement that can be performed on a quantum system is known as a generalized measurement [5,6]. Any measurement on a quantum state is inherently associated with wave function collapse and probability distribution. We recollect the necessary preliminaries briefly below.

Quantum measurements Let $\mathcal{H}$ be a finite dimensional complex Hilbert space, which represents some quantum system. The set of quantum states ρ on $\mathcal{H}$ is denoted by $\mathrm{D}(\mathcal{H})$. A *quantum measurement* on $\mathcal{H}$ is a set $\Lambda \equiv \{\Lambda_x\}_{x\in X} \subseteq \mathrm{L}(\mathcal{H})$ of positive operators indexed by $x \in X$ and satisfies $\sum_x \Lambda_x = \mathbb{1}_{\mathcal{H}}$. Given a quantum state $\rho \in \mathrm{D}(\mathcal{H})$ and a quantum measurement $\Lambda = \{\Lambda_x\}_{x\in X}$, then a probability distribution $p = \{p(x)\}_{x\in X}$ is induced where $p(x) = Tr(\Lambda_x\rho)$ is the probability of the outcome x to occur. In this case, ρ is transformed into the quantum state $\rho_x = \frac{A_x\rho A_x^\dagger}{p(x)}$, where $\Lambda_x = A_x^\dagger A_x$. If $\Pi = \{\Pi_x\}_{x\in X}$ is a set of orthogonal projectors, then the measurement $\{\Pi_x\}_{x\in X}$ is said to be a *von Neumann measurement* [1]. The celebrated Neumark extension theorem [7,8] states that each quantum measurement can be seen as a von Neumann measurement on a larger Hilbert space [9].

We know that in a generalized measurement process, the input state ρ cannot always be retrieved with a nonzero success probability by a "reversing operation" on the state ρ_x. A measurement $\{\Lambda_x\}_{x\in X}$ is called *logically reversible* [10] if the premeasurement state ρ of the measured system is uniquely determined from the postmeasurement state ρ_x and the outcome of the measurement. Ueda et al. in Ref. [10] have shown that the measurement $\{\Lambda_x\}_{x\in X}$ is logically reversible if and only if each measurement operator Λ_x is a reversible operator. Moreover, if for each measurement operator Λ_x, there exists a unitary operator U_x such that

$$U_x\rho_x U_x^\dagger = \rho, \tag{1.1}$$

for each state ρ whose support lies on a subspace $\mathcal{M}$ of $\mathcal{H}$, then $\{\Lambda_x\}_{x\in X}$ is called the *unitarily reversible* measurement [4]. It is clear that any von Neumann measurement $\{\Pi_x\}_{x\in X}$ is not logically reversible except X has only a single element. Note that in a logically reversible measurement, the system's information is preserved during the measurement process. Thus, the reversibility of a measurement is related to the information gained from that measurement. Quantum teleportation [11] can be seen as the problem of reversing a set of quantum operations [4].

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Suppose we are given a logically reversible measurement $\Lambda_u = \{\Lambda_{u,x}\}_{x\in X}$. Since each measurement operator $\Lambda_{u,x}$ is a positive (reversible) operator, then, by the spectral decomposition theorem,

$$\Lambda_{u,x} = \sum_{i\in\Sigma_x} a_x(i)\Pi_x(i), \tag{1.2}$$

where $\sum_{i\in\Sigma_x}\Pi_x(i) = \mathbb{1}_{\mathcal{H}}$ and $a_x(i) > 0$ for any $i \in \Sigma_x$. In particular, if for all $x \in X$ there exist subsets $\{i_s\}_{s=1}^{m_x} \subseteq \Sigma_x$ such that $\sum_{s=1}^{m_x}\Pi_x(i_s)$ are the same projector onto a subspace $\mathcal{M}$ and $a_x(i_1) = \cdots = a_x(i_{m_x})$, then the measurement Λ_u is also a unitarily reversible on the subspace $\mathcal{M}$ [4].

The success probability p_s of reversing, after the measurement with result x, has the upper bound [12,13]

$$p_s \le \frac{\min_{i\in\Sigma_x}\{a_x(i)\}}{p_u(x)}, \tag{1.3}$$

where $p_u(x) = Tr(\Lambda_{u,x}\rho)$. If we define the *total success probability* p_s^{total} of reversing as

$$p_s^{total} = \sum_{x\in X} p_u(x)p_s, \tag{1.4}$$

then

$$p_s^{total} \le \sum_{x\in X}\min_{i\in\Sigma_x}\{a_x(i)\}. \tag{1.5}$$

Note that the above bound is independent of the quantum state ρ.

Shannon and von Neumann entropies A classical state is described by a probability distribution. *Shannon entropy* $H(p)$, for the probability distribution $p = \{p(x)\}_{x\in X}$, is defined by [14]

$$H(p) = -\sum_{x\in X} p(x)\log_2 p(x). \tag{1.6}$$

For a quantum state $\rho \in \mathrm{D}(\mathcal{H})$, the quantum analog of Shannon entropy is *von Neumann entropy*, and is given by

$$S(\rho) = -Tr(\rho\log_2\rho). \tag{1.7}$$

An equivalent expression of $S(\rho)$ is [7],

$$S(\rho) = \min_{\{|\psi_a\rangle, p_a\}} H(\{p_a\}), \tag{1.8}$$

where the minimum is taken over all pure state convex decompositions of ρ. A decomposition minimizes $\{H(\{p_a\}) : \{|\psi_a\rangle, p_a\}\}$ if and only if it is a spectral decomposition of ρ. For an arbitrary ensemble $\{\rho_i, \eta_i\}$, which forms a convex decomposition of ρ, we have

$$S(\rho) \le H(\{\eta_i\}) + \sum_i \eta_i S(\rho_i) \tag{1.9}$$

The equality is achieved if and only if $\{\rho_i\}$ has mutual orthogonal supports.

Quantum discord Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be (the Hilbert spaces of) two quantum systems, $\rho_{AB} \in \mathrm{D}(\mathcal{H}_A\otimes\mathcal{H}_B)$ be a quantum state, ρ_A and ρ_B be the reduced states of ρ_{AB}. In quantum information theory, *quantum mutual information*

$$I_{A:B}(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \tag{1.10}$$

is regarded as a measure of the total correlation [15] between $\mathcal{H}_A$ and $\mathcal{H}_B$. With the quantum conditional entropy, $S(\rho_B|\rho_A) = S(\rho_{AB}) - S(\rho_A)$, quantum mutual information becomes

$$I_{A:B}(\rho_{AB}) = S(\rho_B) - S(\rho_B|\rho_A).$$

Given a von Neumann measurement $\Pi^A = \{\Pi_x^A\}_{x\in X}$ on the quantum system $\mathcal{H}_A$, let us define a conditional entropy on the quantum system $\mathcal{H}_B$ by $S_{B|A}(\rho_{AB}|\{\Pi_x^A\}) = \sum_x \eta_x S(\rho_{B|x})$, where $\rho_{B|x} = \eta_x^{-1}Tr_A(\Pi_x^A\otimes\mathbb{1}_{\mathcal{H}_B}\rho_{AB})$ and $\eta_x = Tr(\Pi_x^A\otimes\mathbb{1}_{\mathcal{H}_B}\rho_{AB})$. Moreover, we denote by

$$\mathcal{J}_{B|A}^{vN}(\rho_{AB}) = S(\rho_B) - \inf_{\Pi^A}\sum_x \eta_x S(\rho_{B|x}), \tag{1.11}$$

which is interpreted as a measure of classical correlation [16,17] between $\mathcal{H}_A$ and $\mathcal{H}_B$. In general, $I_{A:B}(\rho_{AB})$ and $\mathcal{J}_{B|A}^{vN}(\rho_{AB})$ are different, and the difference between them

$$\begin{aligned}\mathcal{D}_A^{vN}(\rho_{AB}) &= I_{A:B}(\rho_{AB}) - \mathcal{J}_{B|A}^{vN}(\rho_{AB}) \\ &= S(\rho_A) - S(\rho_{AB}) + \inf_{\Pi^A}\sum_x \eta_x S(\rho_{B|x}),\end{aligned} \tag{1.12}$$

is called *quantum discord*, which is interpreted as a measure of quantum correlation [16–18]. It is an important *information-theoretic* measure of quantum correlation [19], beyond entanglement measures [20].

Moreover, if we replace the von Neumann measurement in (1.12) with the generalized quantum measurement $M^A = \{M_z^A\}_{z\in Z}$ on $\mathcal{H}_A$ (as described in the Introduction section), then the general quantum discord can be defined as follows:

$$\mathcal{D}_A(\rho_{AB}) = S(\rho_A) - S(\rho_{AB}) + \inf_{M^A}\sum_z \eta_z S(\rho_{B|z}),$$

where $\rho_{B|z} = \eta_z^{-1}Tr_A(\Lambda_z^A\otimes\mathbb{1}_{\mathcal{H}_B}\rho_{AB})$ and $\eta_z = Tr(M_z^A\otimes\mathbb{1}_{\mathcal{H}_{\mathbb{B}}}\rho_{AB})$. Clearly, $\mathcal{D}_A(\rho_{AB}) \le \mathcal{D}_A^{vN}(\rho_{AB})$. Recall that, a *purification* of $\rho \in \mathrm{D}(\mathcal{H}_A)$ is any pure state $|\phi_\rho\rangle\langle\phi_\rho| \in D(\mathcal{H}_A\otimes\mathcal{H}_B)$ such that $Tr_B(|\phi_\rho\rangle\langle\phi_\rho|) = \rho$. It, then, follows from Neumark theorem and the additivity of von Neumann entropy with respect to tensor products, that

$$\mathcal{D}_A(\rho_{AB}) = \mathcal{D}_{AE}^{vN}(\rho_{AB}\otimes|\epsilon_0\rangle\langle\epsilon_0|). \tag{1.13}$$

This paper is organized as follows. Section 2 deals with the construction of a class of logically reversible measurements based on a von Neumann measurement, and provides a relationship between Shannon entropies of the two measurements. Section 3 presents an inequality between quantum discords induced by the two measurements. Conclusion is presented in Section 4.

2. Logically reversible measurements

In this section, we show that it is possible to construct a logically reversible measurement from any given von Neumann measurement, and establish a compact relation between Shannon entropies induced by the two measurements.

Let $\rho \in \mathrm{D}(\mathcal{H})$ and $\Pi = \{\Pi_x\}_{x\in X}$ be a von Neumann measurement with $|X| = n$. Now, based on Π and any $a \in (0, \frac{1}{n})$, we can construct the following logically reversible measurement $\Lambda_u^{(a)} = \{\Lambda_{u,x}^{(a)}\}_{x\in X}$:

$$\Lambda_{u,x}^{(a)} = \{1-(n-1)a\}\Pi_x + \sum_{y\neq x} a\Pi_y. \tag{2.1}$$

The probability distribution $p_u^{(a)} = \{p_u^{(a)}(x)\}_{x\in X}$ is induced, and the probability $p_u^{(a)}(x)$ of the classical outcome x to occur is given by

$$p_u^{(a)}(x) = Tr(\Lambda_{u,x}^{(a)}\rho) = (1-na)p(x) + a, \tag{2.2}$$

where $p(x) = Tr(\Pi_x\rho)$. It is easy to show that the measurement $\Lambda_u^{(a)}$ is not unitarily reversible on any subspace $\mathcal{M}$ with $\dim\mathcal{M} \neq 1$ of $\mathcal{H}$. Note that the total success probability of reversing, after the original von Neumann measurement Π, is zero. However,

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