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Stochastic sensitivity of regular and multi-band chaotic attractors in discrete systems with parametric noise

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ABSTRACT

We consider a response of nonlinear dynamic systems on random disturbances. A stochastic sensitivity of regular and chaotic attractors of discrete systems with parametric noise is studied. Cases of equilibria, cycles, one- and multi-band chaotic attractors are considered, and explicit parametric formulas for the stochastic sensitivity of these attractors are derived. We give a constructive application of this theory to the analysis of the dispersion of random states near chaotic attractors on the example of the logistic map. It is shown how this technique can be effectively used in the analysis of noise-induced crisis bifurcation of merging chaotic bands.

1. Introduction

In nonlinear systems, a wide diversity of dynamic regimes caused by noise is observed. Even weak random disturbances can drastically change system behavior, especially near bifurcation borders. Here, noise-induced transitions [1–3], stochastic resonance [4,5], noise-induced explosions [6,7] and crises [8–10], stochastic bifurcations [11,12] should be mentioned. An analysis of the influence of noise on systems with chaotic attractors is a challenging problem considered in a number of papers [13–18]. As a rule, new stochastic phenomena are discovered by a direct numerical simulation of the stochastic system solutions. However, the next step in studies of the underlying reasons of the found phenomena is connected with the development of the analytical approaches, especially for the parametric investigations.

As the most of noise-induced nonlinear phenomena can be explained by the stochastic transitions between deterministic attractors, or their parts, it is highly important to have an analytical description of the probabilistic distribution of random states near these attractors. An exhaustive description of the probabilistic distributions is given formally by the corresponding Frobenius–Perron equation [19,20] for discrete-time systems, and by the Fokker–Planck–Kolmogorov equation [21] for continuous systems. However, it is technically difficult to use them directly. For example, the Frobenius–Perron equation can be solved analytically only in the specific one-dimensional examples. In these circumstances, constructive asymptotics and approximations are highly relevant [22]. Among them, the stochastic sensitivity function technique is used [23–25]. For discrete systems, this technique was initially elaborated for regular attractors (equilibria and discrete cycles) [26], and later developed for the closed invariant curves [27] and one-band chaotic attractors [28]. Note that chaotic attractors with multiple bands are typical for one-dimensional maps [29,30,3]. It is known that the one-band chaotic attractor is formed as a result of merging bifurcations of disjoint pieces of multi-band chaotic attractors.

In present paper, we extend the sensitivity function technique to the case of discrete systems with multi-band chaotic attractors forced by the parametric noise. Generally, an analysis of the parameter sensitivity is defined as the study of how the output of a model can be attributed to the changes of the system parameters. In present paper, we focus on the study of the sensitivity of the borders of multi-band chaotic attractors to the variation of the noise intensity parameter.

In Section 2, we present a mathematical description of the stochastic sensitivity technique for the fixed deterministic solution, and give explicit formulas for the stochastic sensitivity of stable equilibria and k-cycles.

In Section 3, a theory of the stochastic sensitivity of chaotic attractors is presented. At first, a case of one-band chaotic attractors is investigated, and explicit parametric formulas for the stochastic sensitivity of borders of these attractors are derived. Based on this

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theory, we develop the mathematical technique for the stochastic sensitivity analysis of multi-band chaotic attractors.

The main points of this theory and abilities of its constructive application to the analysis of the dispersion of random states near chaotic attractors are demonstrated in Section 4 for the logistic map. Here, it is also shown how this technique can be used in the parametric analysis of noise-induced crisis bifurcation of merging chaotic bands. Results of our theoretical analysis are compared with the deformation of the probability density functions found by the direct numerical simulation.

2. Stochastic sensitivity of regular attractors

We consider a discrete-time one-dimensional nonlinear system

$$x_{t+1} = f(x_t, \eta_t), \quad \eta_t = \varepsilon \xi_t, \tag{1}$$

where ξ_t is an m-dimensional uncorrelated random process with parameters $E\xi_t = 0$, $E\xi_t\xi_t^\top = V$, V is a covariance $m \times m$-matrix, and ε is a scalar parameter of the noise intensity. Note that other details of the probability distributions of the acting noise are not important.

Consider a solution $\bar{x}_t$ of the corresponding deterministic system (1) with $\varepsilon = 0$ ($\bar{x}_{t+1} = f(\bar{x}_t, 0)$). Let x_t^ε be a solution of the stochastic system (1) with the initial condition $x_0^\varepsilon = \bar{x}_0$. A sensitivity of the solution $\bar{x}_t$ to the random disturbances is defined by the variable

$$z_t = \left.\frac{\partial x_t^\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{x_t^\varepsilon - \bar{x}_t}{\varepsilon}.$$

Dynamics of the pair $(\bar{x}_t, z_t)$ is governed by the stochastic extension system

$$\begin{aligned} \bar{x}_{t+1} &= f(\bar{x}_t, 0) \\ z_{t+1} &= \frac{\partial f}{\partial x}(x_t, 0)\, z_t + \frac{\partial f}{\partial \eta}(x_t, 0)\, \xi_t, \\ \frac{\partial f}{\partial \eta} &= \left(\frac{\partial f}{\partial \eta_1}, \dots, \frac{\partial f}{\partial \eta_m}\right). \end{aligned} \tag{2}$$

The moments $M_t = Ez_t^2$ satisfy the following deterministic system

$$\begin{aligned} \bar{x}_{t+1} &= f(\bar{x}_t, 0) \\ M_{t+1} &= \alpha^2(\bar{x}_t) M_t + s(\bar{x}_t), \end{aligned} \tag{3}$$

where

$$\alpha(x) = \frac{\partial f}{\partial x}(x, 0), \quad s(x) = \frac{\partial f}{\partial \eta}(x, 0)\, V \, \frac{\partial f}{\partial \eta}^\top (x, 0).$$

For the small noise intensity ε, values M_t allow us to approximate a dispersion $D_t = E(x_t^\varepsilon - \bar{x}_t)^2$ of the random states x_t^ε around $\bar{x}_t$: $D_t \approx \varepsilon^2 M_t$.

Using the system (3), one can study the stochastic sensitivity of the different types of attractors, both regular and chaotic.

2.1. Stochastic sensitivity of stable equilibria

Consider a case when the exponentially stable equilibrium $\bar{x}$ is an attractor of the deterministic system (1) with $\varepsilon = 0$. Due to stability of $\bar{x}$, it holds that $|\alpha(\bar{x})| < 1$, and system (3) for $\bar{x}_t \equiv \bar{x}$ has a unique stable stationary solution $M_t \equiv M$, where

$$M = \frac{s(\bar{x})}{1 - \alpha^2(\bar{x})}. \tag{4}$$

The value M is called the stochastic sensitivity of the equilibrium $\bar{x}$.

2.2. Stochastic sensitivity of k-cycles

Let $\bar{x}_1, \dots, \bar{x}_k$ be an exponentially stable k-cycle of the deterministic system (1) with $\varepsilon = 0$. It holds that $\bar{x}_{t+1} = f(\bar{x}_t, 0)$ $(t = 1, 2, \dots, k-1)$, $\bar{x}_1 = f(\bar{x}_k, 0)$. The necessary and sufficient condition of the exponential stability of this k-cycle is $|\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_k| < 1$, where $\alpha_t = \alpha(\bar{x}_t)$. Due to cycle's stability, system (3) has a unique stable k-periodic solution M_t satisfying the equation

$$M_{t+1} = \alpha_t^2 M_t + s_t, \qquad s_t = s(\bar{x}_t). \tag{5}$$

The set $\{M_1, \dots, M_k\}$ defines the stochastic sensitivity of the cycle $\bar{x}_1, \dots, \bar{x}_k$. Here, the element M_1 is a solution of the equation

$$M_1 = [\alpha_1 \cdot \ldots \cdot \alpha_k]^2 M_1 + g_{k+1}, \tag{6}$$

where g_{k+1} is found by iterations

$$g_{t+1} = \alpha_t^2 g_t + s_t, \ \ t = 1, \dots, k, \qquad g_1 = 0.$$

The rest elements $M_2, \dots, M_k$ of the k-periodic solution M_t can be found from the equation (5) recurrently.

For the case of the super-stable k-cycle with $\alpha_1 = \frac{\partial f}{\partial x}(\bar{x}_1, 0) = 0$, we have

$$\begin{aligned} &M_2 = s_1, \ \ M_3 = \alpha_2^2 s_1 + s_2, \ \dots, \\ &M_k = (\alpha_2 \cdot \ldots \cdot \alpha_{k-1})^2 s_1 + (\alpha_2 \cdot \ldots \cdot \alpha_{k-2})^2 s_2 + \ldots + s_{k-1}, \\ &M_1 = (\alpha_2 \cdot \ldots \cdot \alpha_k)^2 s_1 + (\alpha_2 \cdot \ldots \cdot \alpha_{k-1})^2 s_2 + \ldots + s_k. \end{aligned}$$

3. Stochastic sensitivity of chaotic attractors

Consider a case when the deterministic system (1) with $\varepsilon = 0$ has a chaotic attractor. Here, along with the case of one-band chaotic attractors, we will study the stochastic sensitivity of more complicated, multi-band chaotic attractors.

3.1. Stochastic sensitivity of the one-band chaotic attractors

First consider a case of the one-band chaotic attractor. This means that the chaotic attractor $\mathcal{A}$ is a single interval: $\mathcal{A} = [a, b]$. Let the function $f(x, 0)$ have a single maximum (see Fig. 1a) at the point $c \in (a, b)$:

$$\max_{[a,b]} f(x, 0) = f(c, 0), \quad \frac{\partial f}{\partial x}(c, 0) = 0.$$

One can connect borders a and b of the attractor $\mathcal{A}$ with the point c:

$$b = f(c, 0), \qquad a = f(b, 0) = f(f(c, 0), 0).$$

The stochastic sensitivity of the borders a and b of the attractor $\mathcal{A}$ is defined by the stochastic sensitivity of the corresponding points of the solution of the deterministic system (1) with $\varepsilon = 0$, passing through these borders. The simplest variant of this solution is $\bar{x}_t$ with the initial state $\bar{x}_1 = c$. Indeed, the first iterations give us $\bar{x}_2 = f(\bar{x}_1) = b$, $\bar{x}_3 = f(\bar{x}_2) = a$. Note that the subsequent states of this solution belong to the (a, b), and never fall to borders $x = a$ and $x = b$. In these circumstances, the stochastic sensitivity of the right border $x = b$ of the attractor $\mathcal{A}$ coincides with the stochastic sensitivity M_2 of the state $\bar{x}_2$ of the considered solution $\bar{x}_t$: $M(b) = M_2$. Similarly, for the stochastic sensitivity $M(a)$ of the left border, we have $M(a) = M_3$, where M_3 is the stochastic sensitivity of M_3. It follows from the general system (3) that

$$M_2 = \alpha_1^2 M_1 + s_1, \ \ M_3 = \alpha_2^2 M_2 + s_2, \tag{7}$$