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Level rearrangement in exotic-atom-like three-body systems

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ABSTRACT

We study systems of three bosons bound by a long-range interaction supplemented by a short-range potential of variable strength. This generalizes the usual two-body exotic atoms where the Coulomb interaction is modified by nuclear forces at short distances. The energy shift due to the short-range part of the interaction combines two-body terms similar to the ones entering the Trueman–Deser formula, and three-body contributions. A sudden variation of the energy levels is observed near the coupling thresholds of the short-range potential. But the patterns of rearrangement are significantly modified as compared to the two-body case.

1. Introduction

Exotic atoms have a long history, and have stimulated interesting developments in the quantum dynamics of systems involving both long-range and short-range forces. For the exploratory studies presented here, the refinements of effective theories [1] are not required, and we shall restrict ourselves to the Schrödinger framework, as reviewed, e.g., in [2] for the three-dimensional case, and extended in [3] for the two-dimensional one.

In units simplifying the treatment of the pure Coulomb case, where the strength becomes 1 and the reduced mass 1/2, an exotic atom can be modeled as

$$-\Delta\Psi + V\,\Psi = E\,\Psi\,, \quad V = -\frac{1}{r} + \lambda\, v(r)\,, \tag{1}$$

where r is the inter-particle distance and $\lambda\, v(r)$ the short-range correction, with a variable strength for the ease of discussion.

In most actual exotic atoms, there is a strong absorptive component in the short range interaction, so one has to use either a complex (optical) potential v or a coupled-channel formalism. Probably, the $(\bar{D}_s,\, p)$ atom, with a proton and an anticharmed meson $\bar{D}_s = (\bar{c}s)$ of charge -1 and strangeness -1, escapes any absorption, since it lies below any threshold such as $(\bar{D}^0(\bar{c}u)\,\Lambda(sud))$, but it is not yet accessible experimentally. In this study, we make the somewhat drastic simplification of a purely real short-range term v.

When the above spectral problem is solved, the most striking observations are:

1. The energy shift $\delta E = E - E_n$, as compared to the pure-Coulomb energies $E_n = -1/(4n^2)$, is often rather small, but is usually not given by ordinary perturbation theory: for instance, an infinite hard-core of small radius corresponds to a small energy shift but to an infinite first order correction.
2. Each energy $E(\lambda)$, as a function of the strength parameter λ, is almost flat in a wide interval of λ, with a value close to some E_n, one of the pure Coulomb energies. For S-wave states δE is well approached by a formula by Deser et al., and Trueman [4],
$$\delta E \simeq \frac{a}{2n^3}\,, \tag{2}$$
where a is the scattering length in the short-range interaction $\lambda\, v(r)$ alone.
3. If $v(r)$ is attractive, when the strength λ approaches one of the positive critical values at which a first or a new bound state appears in the spectrum of $\lambda\, v$ alone, the energy $E(\lambda)$ quits its plateau and drops dramatically from the region of atomic energies to the one of deep nuclear binding. It is rapidly replaced in the plateau by the next level. This is known as *level rearrangement* [5]. Note that level rearrangement disappears if absorption becomes too strong [2,6].
4. The above patterns are more general, and hold for any combination of a long-range and a short-range interaction, say $V = V_0 + \lambda\, v(r)$, as encountered, e.g., in the physics of cold atoms where a long-range confining interaction is supple-

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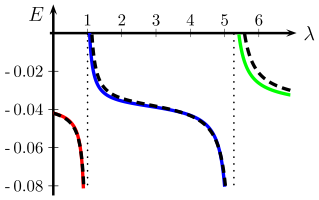

Fig. 1. (Color online.) Level rearrangement for a two-body system bound by the potential (4). The dashed line indicates the approximation corresponding to the Deser–Trueman formula (3). The dotted vertical lines show the coupling thresholds for the short-range part of the interaction.

mented by the direct interaction among the atoms [7]. The Deser–Trueman formula of S-states is generalized as

$$\delta E \simeq 4\pi\, |\Psi_0(0)|^2\, a\,, \tag{3}$$

where Ψ_0 is the normalized wave-function in the external potential V_0. If V_0 supports only one bound state, then the rearrangement "extracts" states from the continuum, instead of shifting radial excitations that are already bound. An illustration is given in Fig. 1, with a superposition of two exponential potentials of range parameters $\mu = 1$ and $\mu = 100$, namely

$$\begin{aligned} V(r) &= 2\, v_e(1,r) + \lambda\, v_e(100,r)\,, \\ v_e(\mu,r) &= -1.4458\, \mu^2\, \exp(-\mu\, r)\,, \end{aligned} \tag{4}$$

where $v_e(\mu, r)$ is tuned to start binding at unit strength.

5. There are several possible improvements and alternative formulations of (3). For instance, a, the bare scattering length in $\lambda\, v(r)$ can be replaced by the "long-range corrected" scattering length where the solutions of the radial equation for $\lambda\, v(r)$ are matched to the eigenfunction of the external potential. See, e.g., [8], and refs. there. In the physics of cold atoms, one is more familiar with the approach by Busch et al. [7]. It deals with the case of an harmonic oscillator modified at short distances, but the derivation can be generalized as follows. Let $u(E,r) = \sqrt{4\pi}\, r\, \Psi$ the s-wave reduced radial wavefunction for $V_0(r)$ that is regular at large r, at energy $E < 0$, with some normalization, e.g., $u(E,r)\, \exp(r\sqrt{-E}) \to 1$ for $r \to \infty$. The levels in V_0 correspond to the quantization condition $u(E_0, 0) = 0$, where E_0 is an eigenenergy of V_0, for instance the ground state. When a point-like interaction of scattering length a is added, then the boundary condition is modified into

$$u(E,0) + a\, \partial_r u(E,0) = 0\,, \tag{5}$$

which can be expanded near E_0 to give

$$\delta E = E - E_0 \simeq -a\, \frac{\partial_r u(E_0,0)}{\partial_E u(E_0,0)}\,. \tag{6}$$

Now, the equivalence of (6) and (3), rewritten as $\delta E \simeq a\, [\partial_r u_N(E_0,0)]^2$, where u_N is the normalized version of u, comes from the relation

$$\partial_r u(E_0,0)\, \partial_E u(E_0,0) = -\int_0^\infty u(E_0,r)^2\, \mathrm{d}r\,, \tag{7}$$

which is easily derived from the Wronskian identity, widely used in some textbooks [9], here applied to energies E and E_0.

6. The generalization to a number of dimension $d \neq 3$ is straightforward for $d > 3$. For $d = 1$, the first plateau is avoided, as the short-range potential, if attractive, develops its own discrete spectrum for any $\lambda > 0$. The case of $d = 2$ is more delicate: see, e.g., [6,10].

Our aim here is to present a first investigation of the three-body analog of exotic atoms. There are already studies of systems such as (K^-, d), where $d = (p, n)$ is the deuteron, in which the neutron feels only the short-range part of the interaction. We will study systems in which the three constituents are already bound by the long-range component of the potential. This is the first attempt, at least to our knowledge. We consider three identical bosons, relevant for three atoms in a confining trap.[1] We have in mind less symmetric systems for future work. We address the following questions: Is there a pattern similar to the level rearrangement? Is there a generalization of the Deser–Trueman formula? What are the similarities with the case where the long-range interaction is replaced by an overall harmonic confinement?

Note that the occurrence of plateaus and sudden drops of the energies as a function of the coupling strength is not very usual, as these energies are monotonic and concave functions of any parameter entering linearly the Hamiltonian [11]. For instance, in a pioneering study of three-boson energies, Osborn [12] fund some type of rearrangements in the three-body spectrum corresponding to a Yukawa interaction, but it was later acknowledged that this calculation suffers from some numerical instability, as the computed Faddeev energies violate a rigorous lower bound [12,13]. Unfortunately, the erroneous plot was reproduced in a seminal textbook on the three-body problem [14].

The paper is organized as follows. In Sec. 2, we give some basic reminders about the spectrum of a three-boson systems from the Borromean limit of a single bound state to the regime of stronger binding, with a word about the numerical techniques. The results corresponding to a superposition $V_0(r) + \lambda\, v(r)$ are displayed in Sec. 3. An interpretation is attempted in Sec. 4, with a three-body version of the Deser–Trueman formula. Section 5 is devoted to our conclusions.

2. The three-boson spectrum with a simple potential

If two bosons interact through an attractive potential, or a potential with attractive parts, $\lambda\, v_0(r)$, a minimal strength is required to achieve binding, say $\lambda > \lambda_2^{\rm cr}$. A collection of values of $\lambda_2^{\rm cr}$ can be found, e.g., in the classic paper by Blatt and Jackson [15]. In the following, we shall normalize v_0 so that $\lambda_2^{\rm cr} = 1$.

If one assumes that $v_0(r)$ is attractive everywhere, once two bosons are bound, the 3-boson system is also bound. (The case of potentials with a strong inner repulsion would require a more detailed analysis which is beyond the scope of this preliminary investigation.) This means that for a single monotonic potential

$$V_0 = \lambda\, [v_0(r_{12}) + v_0(r_{23}) + v_0(r_{31})]\,, \tag{8}$$

where $r_{ij} = |\boldsymbol{r}_j - \boldsymbol{r}_i|$, the minimal coupling to achieve three-body binding, $\lambda = \lambda_3^{\rm cr}$, is less than 1, in our units. This is implicit in the seminal paper by Thomas [16]. The inequality $\lambda_3^{\rm cr} < \lambda_2^{\rm cr}$ is now referred to as "Borromean binding", after the study of neutron halos in nuclear physics [17]. One gets typically $\lambda_3^{\rm cr} \simeq 0.8$, i.e., about 20% of Borromean window [18].

In short, the three-boson spectrum has the following patterns:

- For $\lambda < \lambda_3^{\rm cr}$, no binding.
- For $\lambda_3^{\rm cr} < \lambda < 1$, a single Borromean bound state, and, for λ close to 1, a second three-body bound state just below the two-body energy.
- Very near $\lambda = 1$, the very weakly bound Efimov states.
- For $1 < \lambda$, two bound states below the $2 + 1$ break-up, and further bound states when λ becomes very large.

[1] At least for harmonic confinement, an external potential can be rewritten as a sum of pair interactions.