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Two-dimensional electrostatic model for the Van der Pauw method

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A B S T R A C T

We use a two-dimensional electrostatic model to explore Van der Pauw (VdP) method on samples with holes. Results of electrostatic simulation based on the model are consistent with the experimental results. The quantitative investigation of VdP method on sample with one hole indicate the value of f, i.e. the right hand side of VdP equation, is depend on the position of contacts and the size of holes. We also quantitatively explore the VdP method used on sample with two holes in special position of contacts and sample geometry. According to the guidance of simulation and experiment, we find proper position of contacts, which make VdP equation $f = 1$ valid again on sample with holes. It is helpful for the optimization of VdP method and measuring the resistivity of sample with holes.

1. Introduction

The Van der Pauw (VdP) method is widely used in measurements of resistivity [1–3]. This method can also be used to investigate the Hall effect [1,4] and heat transport measurement [5]. Four point-like contact points are put on the circumference of the singly connected sample of arbitrary shape, isotropic resistivity small and homogeneous thickness, which is the ideal condition of VdP method. The relation between two directly measured resistances R_1, R_2, thickness d and resistivity ρ obeys famous VdP equation:

$$f \equiv e^{-\frac{\pi R_1 d}{\rho}} + e^{-\frac{\pi R_2 d}{\rho}} = 1, \tag{1}$$

which is dexterously proved by using two-dimensional conformal mapping [1]. The VdP equation can also be derived by solving Laplace's equation, $\nabla^2\varphi = 0$, in three-dimensional Cartesian coordinate by using separation of variables and the influence of non-zero thickness is shown in analytic form [6].

There are many investigations about the non-ideal condition of VdP method. The investigation of samples with anisotropic resistivity medium is done both experimentally and theoretically [7]. The influence of non-ideal point contact and the distance between contact points and circumference is investigated both experimentally [8] and numerically [9]. In ref. [10], a strange phenomenon is explored that VdP equation becomes an inequality

$$f \leq 1, \tag{2}$$

on the condition that the sample has an isolated hole. Experimentally, they find that when the contacts become close to each other, the measurement results are consistent with Eq. (1); theoretically, they predict the minimum value of f. From mathematical point of view, the Riemann theorem of complex analysis predicts the necessity of the change of Eq. (1). The reason is that a doubly connected region can not be mapped into a singly connected region by using conform mapping [11]. VdP method can experimentally determine an important geometrical parameter, the Riemann modulus, of a sample with an isolated hole [12]. However, there is no general theory and rigorous proof of inequality (2). The invalidity of VdP equation (1) results in some difficulties of resistivity measurement. How to use VdP method on sample with an isolated hole or even more complex condition is an open and meaningful subject.

In this paper, after a concise introduction of electrostatic model and simulation procedure, we make a series of electric field simulations based on the model. We then use experimental and simulation method to explore VdP method on samples with one hole and two holes. The comparison of these results indicates that the simulation method based on the two-dimensional electrostatic model can accurately estimate the value of f. We investigate the correlation between the value of f and the position of four contacts on sample with one hole and two holes. In the last section, we summarize our work.

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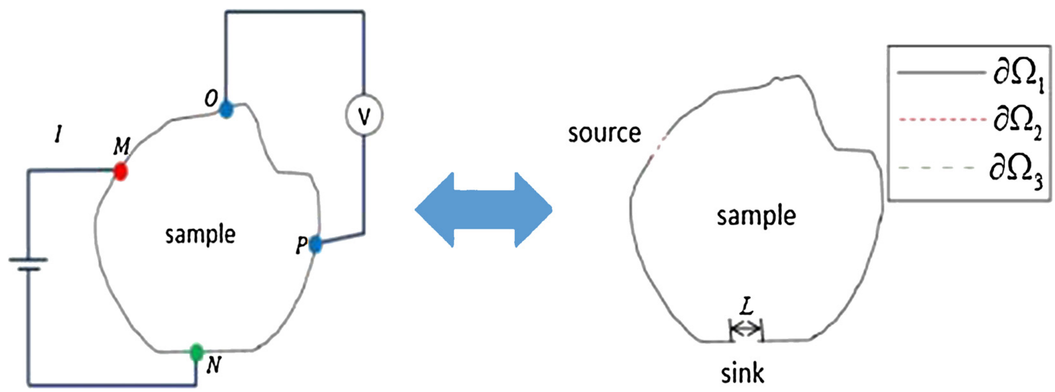

Fig. 1. The corresponding relation between experimental condition and boundary condition. The left part of this figure is real experimental condition and the sample is three-dimensional. The right part of this figure is the corresponding electrostatic model. Based on the assumption, it is two-dimensional. L is the length of contact area and it is magnified for clearness and both of real experiment and simulation require L is extremely small for point-like contact.

2. Electrostatic model

The electrostatic model is based on the following assumptions

(i) The z-direction is perpendicular to the sample plate. The thickness of the sample is d. There is no change of voltage and current flow in the z-direction because d is sufficiently small. Thus

$$\frac{\partial \varphi}{\partial z} = 0, \quad \text{where } z \in [0, d]. \tag{3}$$

(ii) Two current contacts can be regarded as source and sink of the electric field. The Neumann-type boundary conditions

$$\frac{\partial \varphi}{\partial \vec{n}} = \begin{cases} 0, & \text{on } \partial\Omega_1 \\ \vec{E}, & \text{on } \partial\Omega_2 \\ -\vec{E}, & \text{on } \partial\Omega_3 \end{cases} \tag{4}$$

is satisfied. φ is the voltage and the boundary $\partial\Omega_1$ $\partial\Omega_2$ and $\partial\Omega_3$ are shown in Fig. 1.

(iii) The basic condition of VdP method, except for singly connected sample, are satisfied. It is worth to stress that the resistivity of the sample is isotropic and the contacts are ohmic.

Combining the equation of continuity obeyed current density and the differential form of Ohms Law, we find that the distribution of electric potential must satisfy Laplace's equation

$$\nabla^2 \varphi = 0. \tag{5}$$

The left part of Fig. 1 represents real experimental condition. The sample is three-dimensional. There are four points M, N, O, P on the circumference of the sample. We put two current probes at point M and N, i.e. let the current I enter the system at point M and leave at point N. In order to measure the voltage, we put two probes at point O and P. Considering assumption (i), we can use a two-dimensional electrostatic model to explore VdP method. The distribution of voltage can be obtained by solving a definite solution problem consisted of two-dimensional Laplace equation (5) and boundary condition Eqs. (3) and (4). The schematic diagram of the equivalent model is shown in the right part of Fig. 1. Note that the size of source and sink is magnified for clearness.

3. Simulation and qualitative investigation

In general, it is difficult to derive the analytical solution of the electrostatic model because of the complexity of the boundary conditions. But we can obtain the numerical solution by using the MATLAB PDE toolbox. After solving the distribution of voltage φ, we can directly obtain $\Delta\varphi$, which is the difference of voltage between P and O. The contact area is

$$S = Ld \tag{6}$$

where L is the length of the current contacts, which is shown in Fig. 1. Combining $I = JS$, where J is the current density, we can use the simulation quantities $\Delta\varphi$, L and J to derive R_1

$$R_1 = \frac{\Delta\varphi}{JLd}. \tag{7}$$

Similarly, we can derive R_2. Finally, we use $f \equiv e^{-\frac{\pi R_1 d}{\rho}} + e^{-\frac{\pi R_2 d}{\rho}}$ to calculate the value of f. Note that combining the differential form of Ohms Law and Eq. (7), we find out that the calculation of f is independent of resistivity ρ. As long as the sample is singly connected, $f \equiv 1$ is valid for all samples with different resistivity. Ref. [10] also points out in condition that the sample has one hole and the four contacts are put on the outer boundary, the validity of VdP inequality (2) should be independent of the resistivity of the sample.

The simulation of a rectangular sample without hole is performed. The position of the contact points M, N, O, P and the distribution of voltage and electric field strength on the sample region is shown in Fig. 2(a). Simulations of sample with arbitrary shape are also performed. All results are consistent with Eq. (1). The simulation of the rectangular sample with an isolated hole is also performed, where the boundary condition of the inner boundary (the second boundary) is the same as $\partial\Omega_1$ in Eq. (7). The position of four points doesn't change but the results are consistent with inequality (2). The distribution of voltage and electric field strength on the sample is shown in Fig. 2(b). The comparison between Fig. 2(a) and Fig. 2(b) implies that the existence of inner boundary disturbs the original field lines in order to satisfy the boundary condition.

We then explore VdP method on sample with two holes. The four contacts are put on the outer boundary. The simulation indicates the distribution of voltage and electric field strength on the sample is shown in Fig. 3. In Fig. 4 the black line is the fitting curve of VdP method on singly connected sample and the red circle points are the experimental results of the sample with two holes. Both of the simulation and experimental results indicate that when four contacts are put on the outer boundary of the sample with two holes, f still obey VdP inequality (2). The comparison between Fig. 3 and Fig. 2(a) also reveals the disturbance of holes to the electric field lines. We predict that VdP inequality (2) is valid not only on sample with one hole or two holes, but on sample with many (three, four and more) holes in the case of outer boundary contacting.