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Permanence and boundedness of solutions of quasi-polynomial systems

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ABSTRACT

In this paper we consider analytical properties of a class of general nonlinear systems known as Quasi-Polynomial systems. We analyze sufficient conditions for permanence and boundedness of solutions, and illustrate our approach with an application to Lamb equations describing the evolution of electromagnetic modes in an optical maser.

1. Introduction

In recent years the authors analyzed several analytical properties of Quasi-Polynomial (QP) dynamical systems. For example, the problem of stability in the sense of Lyapunov [1], was the subject of references [2–5]. In [6] integrability properties of QP systems were analyzed through associated bi-linear non-associative algebras, and in [7] it was shown that any quasi-polynomial invariant of a QP system is related to a similar invariant of a Lotka–Volterra (LV) dynamical system. Applications to systems of biological interest was presented in [8]. More recently, in [9] we presented a connection between asymptotic stable interior fixed points of square ($m = n$), or isomonomial, QP systems and evolutionary stable states, a concept of evolutionary games.

The QP systems are defined as follows:

$$\dot{x}_i = l_i x_i + x_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}}; \quad i = 1, \ldots, n. \tag{1}$$

Here $x_i \in \Re^n$, with A and B real, constant rectangular matrices and $m \geq n$ is assumed [10]. This class of systems encompass many systems of interest [10–12] and was extensively studied in literature [2–13].

One interesting property is the possibility to map Eq. (1) into a quadratic Lotka–Volterra (LV) system [10]. Define new variables U_α, $\alpha = 1, \ldots, m$, satisfying:

$$x_i = \prod_{\beta=1}^{m} U_\beta^{D_{i\beta}}, \tag{2}$$

with D an invertible matrix with components $D_{i\beta}$. Now choose $D = \tilde{B}^{-1}$, where:

$$\tilde{B} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} & b_{1,n+1} & \cdots & b_{1,m} \\ B_{21} & B_{22} & \cdots & B_{2n} & b_{2,n+1} & \cdots & b_{2,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mn} & b_{m,n+1} & \cdots & b_{m,m} \end{bmatrix}. \tag{3}$$

Parameters b_{jk} are arbitrary provided $\tilde{B}$ is invertible and reasons for its introduction can be found in [10–12]. By defining the auxiliary variables $x_k(t = 0) = 1$ for $k = n + 1, \ldots, m$, we have that $U_\alpha = \prod_{i=1}^{m} x_i^{D^{-1}_{\alpha i}}$ with $D = \tilde{B}^{-1}$. The variables U_α then satisfy the system of equations:

$$\dot{U}_\alpha = (\tilde{B}\tilde{l})_\alpha U_\alpha + U_\alpha \sum_{\beta=1}^{m} (\tilde{B}\tilde{A})_{\alpha\beta} U_\beta; \quad \alpha = 1, \ldots, m, \tag{4}$$

which is a quadratic LV type system in the variables U_α. Here

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$$\tilde{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad (5)$$

and

$$\tilde{l} = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (6)$$

It is usual to define the $m \times m$ matrix $M \equiv BA$ and $\lambda \equiv Bl$. Note that:

$$BA = \tilde{B}\tilde{A}, \ \ Bl = \tilde{B}\tilde{l}.$$

Considering that $\prod_{\beta=1}^{m} U_{\beta}^{\tilde{B}^{-1}_{\alpha\beta}} = 1; \alpha = n+1, \ldots, m$ the dynamics of the n dimensional system (1) takes place in a manifold of the $m \geq n$ dimensional LV space. Further details on the properties of this mapping can be found in [10–12]. In [12] a generalization is presented that encompasses systems not originally in the QP format.

As shown in the above cited references, the LV canonical format is useful to analyze analytical properties of QP systems. In this paper we extend the results presented in [2–9] by considering sufficient conditions for uniformly bounded solutions and permanence of trajectories, which we present in the following section.

2. Permanence and uniformly bounded solutions

Let us consider the criteria for permanence in dynamical systems. The differential equations $\dot{x}_i = f_i(x_1, \ldots, x_n)$ is said to be permanent if there exists a constant $\delta > 0$ such that, if $x_i(0) > 0$, $\forall i = 1, \ldots, n$, then $\liminf_{t\to\infty}[x_i(t)] > \delta$, $i = 1, \ldots, n$, where δ does not depend on the initial condition $x_i(0)$ [14], see also section 12.2, [15]. For a permanent system, the boundaries of the phase space are repellents. On the other side, systems with bounded solutions obey $\limsup_{t\to\infty}[x_i(t)] \leq d_i$ for some constants d_i. A permanent system is linked to the existence of an Average Lyapunov Function [15]. We address the question of how the solutions of a QP system relates to permanent or bounded solutions in its associated LV system. We state the following theorem:

Theorem 1. *Considering a LV system with bounded solutions, then the following properties hold for the associated QP system:*

1. *If $\tilde{B}^{-1}_{ij} \geq 0$ for all $i = 1, \ldots, n$, the solutions of the corresponding QP system are bounded.*
2. *If $\tilde{B}^{-1}_{ij} \leq 0$ for $i = 1, \ldots, n$, the solutions of the corresponding QP system are permanent.*

Proof. To prove Theorem 1, note that, if the orbits in the LV system are bounded then there exists some $R_i > 0$ such that for all $t > 0$ and all i we have:

$$U_j(t) \leq R_j, \ \ R_j > 0 \, \forall j = 1, \ldots, m.$$

If $\tilde{B}^{-1}_{ij} \geq 0$ then

$$U_j(t) \leq R_j \Rightarrow U_j^{\tilde{B}^{-1}_{ij}}(t) \leq R_j^{\tilde{B}^{-1}_{ij}}.$$

We then have that

$$x_i(t) = \prod_{j=1}^{m} U_j^{B^{-1}_{ij}}(t) \leq \prod_{j=1}^{m} R_j^{B^{-1}_{ij}} \equiv \Delta_i,$$

for $\Delta_i > 0$. Thus $x_i(t) \leq \Delta_i$ for all $t > 0$.

On the other hand if $\tilde{B}^{-1}_{ij} \leq 0$ we have

$$U_j(t) \leq R_j \Rightarrow U_j^{\tilde{B}^{-1}_{ij}}(t) \geq R_j^{\tilde{B}^{-1}_{ij}},$$

which implies in

$$x_i(t) = \prod_{j=1}^{m} U_j^{B^{-1}_{ij}}(t) \geq \prod_{j=1}^{m} R_j^{B^{-1}_{ij}} = \Delta_i.$$

Thus $x_i(t) \geq \Delta_i$ for all $t > 0$.

It is important to note that, since $\tilde{B}$ is invertible, then in any row of $\tilde{B}^{-1}$ there is at least one non-null element, and the inequalities used above are always satisfied. This finishes the proof of the theorem. □

Permanent LV systems possess a unique interior fixed point and $(-1)^n \det M > 0$. QP systems with $m > n$ usually are mapped into a LV system with $\det M = \det BA = 0$ [4,5]. In order to have $\det M \neq 0$ we now restrict ourselves here to the case of square QP systems ($m = n$). In this case we have $\tilde{B} = B$ and $\tilde{A} = A$:

Theorem 2. *Given a square QP system, if its associated LV system is permanent, then:*

1. *If $B^{-1}_{ij} \geq 0$ for all $i = 1, \ldots, n$, the solutions of the corresponding square QP system are permanent.*
2. *If $B^{-1}_{ij} \leq 0$ for all $i = 1, \ldots, n$, the solutions of the corresponding square QP system are bounded.*

Proof. Let the solutions of the LV system to be permanent, then for every $t > 0$:

$$U_j(t) \geq d_j, \ \ d_j > 0 \, \forall j = 1, \ldots, n.$$

If $B^{-1}_{ij} \geq 0$ then

$$U_j(t) \geq d_j \Rightarrow U_j^{B^{-1}_{ij}}(t) \geq d_j^{B^{-1}_{ij}}.$$

This in turn implies in

$$x_i(t) = \prod_{j=1}^{n} U_j^{B^{-1}_{ij}}(t) \geq \prod_{j=1}^{n} d_j^{B^{-1}_{ij}} = \delta_i.$$

Therefore exists $\delta_i > 0$ such that $x_i(t) \geq \delta_i$ for all $t > 0$. When $B^{-1}_{ij} \leq 0$ we have

$$U_j(t) \geq d_j \Rightarrow U_j^{B^{-1}_{ij}}(t) \leq d_j^{B^{-1}_{ij}}.$$

Thus

$$x_i(t) = \prod_{j=1}^{n} U_j^{B^{-1}_{ij}}(t) \leq \prod_{j=1}^{n} d_j^{B^{-1}_{ij}} = \delta_i.$$

Therefore $x_i(t) \leq \delta_i$ for all $t > 0$ and this finishes the proof. □

According to [15], for a permanent LV system there exists $\delta > 0$ such that $\delta < \liminf_{t\to\infty}[x_i(t)]$, $\forall i$. In this case there is also a constant R such that $\limsup_{t\to\infty}[x_i(t)] \leq R$, $\forall i$ provided $(x_1, \ldots, x_n) \in$