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Effect of stochastically moving border on basins of attraction in a class of piecewise smooth maps

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A B S T R A C T

Determination of the basin of attraction of an attractor plays an important role in understanding the dynamics of a system. In all the existing literature, the basin of attraction of any attractor has been described to be deterministic. In this paper we show the existence of a non-deterministic basin of attraction of an attractor. We have considered a piecewise smooth (PWS) one-dimensional map, having stochastically varying border, which is allowed to move in a small bounded region of the phase space while retaining the deterministic dynamics on each compartment of the phase space. In case of this type of systems there exists a region in the phase space with the property that orbits starting from a single point lying inside this region do not display the same property of convergence or divergence, i.e., one may converge while another may diverge. In other words, the convergence or divergence of an orbit starting from a point inside this region is a probabilistic event, and the probabilities of convergence and divergence are both non-zero. We also derive the upper and lower bounds of the corresponding probability curves. Since all physical systems contain noise, the occurrence of such non-deterministic basin of attraction is a definite possibility, if the noise affects the position of the border. This may lead to dangerous consequences, as a region of the basin of attraction of an attractor may become non-deterministic, with a non-zero probability of divergence of orbits starting inside it.

1. Introduction

Piecewise smooth maps (PWS) are frequently used to model many physical and engineering systems [1–6]. In an n dimensional piecewise smooth map the phase space is divided into two or more compartments separated by $(n-1)$ dimensional subspaces called 'border'. The map is not differentiable at the 'border'. The map may be either continuous (PWS continuous map) or discontinuous (PWS discontinuous map) at the border.

Piecewise smooth continuous maps are used to model many systems in electrical engineering (DC-DC converter) [7,8], mechanical engineering (hybrid impacting systems) [4], biology[9], economics, etc. Likewise many systems are also modelled by PWS discontinuous maps [10–13]. In these systems described by PWS maps, as a system parameter is varied, a fixed point may move towards the border and at a certain critical value of the parameter, it may collide with the border to give rise to a different kind of attractor. This kind of bifurcation is called border collision bifurcation [1,14,11,15,16]. To study the dynamics of such PWS maps, one generally uses the border-collision normal from [11,16]. The border-collision normal form of such systems is obtained by linearizing the functional form of the PWS map in a sufficiently small neighbourhood of the border.

These physical systems always exist in noisy environments, and therefore it is necessary to study the dynamics of such systems taking into account the effect of various possible noise terms that can significantly affect the system dynamics. Very few such studies have been reported in the literature. Griffin and Hogan showed that the bifurcation point may change due to additive noise in one dimensional piecewise smooth discontinuous maps [17]. Simpson studied the invariant density of two dimensional piecewise smooth continuous maps with singularity in its domain, in presence of a small-amplitude additive Gaussian noise [18]. In these studies the noise term appears in the functional form of the map, and hence it affects only the subsequent iterates but does not affect the position of the border.

However this assumption may not always be true and there may arise situations where the border is not fixed due to the presence of some special kind of noise in the system [19]. Such variation of the border may be caused due to mechanical reasons (loosely fitted panels or dividers), due to variations in the environment (effects of change of pressure or temperature), due to some defined rule of measurements (e.g., if in the data measurement

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technique a transition occurs if the observation is within some tolerance level of a theoretical switching surface). In these cases we have to consider a border moving stochastically, in a bounded region of the phase space. Glendinning studied the attractor of such a system and showed that if the map without the noise component is a contraction mapping with a stable fixed point, then the attractor of such a system is the attractor of an iterated function system. In absence of such conditions some other possibilities of the dynamics has also been investigated numerically [19]. Besides this, Simpson and Kuske [20] studied the regular grazing bifurcations of impacting systems with stochastic perturbations appearing from impact events. By assuming that only the impacting dynamics are affected by coloured noise, they constructed stochastic versions of the Nordmark map, and analysed three different cases where noise appears in different nonlinear or multiplicative ways depending upon the source of noise.

The basin of attraction of an attractor is defined as a region of the phase space, with the property that any orbit which starts inside this region, ultimately converges to the attractor. Moreover if an orbit starts from any point outside this basin of attraction, then it certainly does not converge to that attractor; it either converges to some other attractor or simply diverges. All points in the phase space belong to a unique basin of attraction. In this paper we show that the above situation may not hold true for piecewise smooth maps with stochastically moving border. In case of such maps there exists a region in the phase space such that two orbits starting from a point inside that region may behave differently—one may converge while the other may diverge.

In this paper we have considered a 1-dimensional PWS continuous map with stochastically varying border, maintaining deterministic dynamics in the two compartments separated by the border. The map is contracting (having slope between 0 and 1) in the left side of the border, and expanding (having slope greater than one) in the right side. This types of maps, in absence of noise, exhibit a stable fixed point on the left hand side of the border and an unstable fixed point on the right hand side of the border for some value of the bifurcation parameter. The basin of attraction of the stable fixed point is the interval from $-\infty$ to the unstable fixed point of such a map. However if we consider random variation of the border in a small bounded region about its deterministic location (this region does not contain the stable fixed point), then the border collision normal form becomes a PWS discontinuous map. If the unstable fixed point lies within the domain of variations of the border for some value of the parameter, there exists a non-deterministic basin of attraction. In this case, convergence or divergence of orbits starting from points in this region will have non-zero probabilities. We have also derived the upper and lower bounds of the corresponding probabilities and have verified the results by numerical simulation.

2. Mathematical formulation

Suppose we have a one dimensional continuous PWS map and the border-collision normal form of the map is given by

$$x_{n+1} = \begin{cases} \mu + a x_n & : x_n \le 0 \\ \mu + b x_n & : x_n \ge 0 \end{cases} \tag{1}$$

where $0 < a < 1$ and $b > 1$. When $\mu < 0$ there exist two fixed points, x_L to the left hand side of the border and x_R to the right hand side of the border. Here

$$x_L = \frac{\mu}{1-a}\,; \quad x_R = \frac{\mu}{1-b}$$

We notice that x_R is an unstable fixed point of (1) whereas x_L is a stable fixed point of (1), with basin of attraction $(-\infty, x_R)$.

Next we assume that the position of the border is not deterministic, due the presence of some noise in the system. At every iteration, the border moves in a small neighbourhood $(-\delta, \delta)$ containing $x = 0$, position of the border in the noiseless deterministic case. Then the border collision normal form of the system (1) becomes a PWS discontinuous map given by

$$x_{n+1} = \begin{cases} \mu + a x_n & : x_n \le \varepsilon_n \\ \mu + b x_n & : x_n > \varepsilon_n \end{cases} \tag{2}$$

where ε_n denotes the stochastic noise term. Consider that ε_n's are independent and follow a particular probability distribution inside the interval $(-\delta, \delta)$. Without loss of generality, in the above equation (2) we have included the equality to the left hand side, however it does not matter if one considers the equality to be included on the other side.

Before we proceed, let us clarify the reason for considering the mathematical form (2) for our analysis. Due to the effect of noise, the border is considered to move in a very small neighbourhood about its position in absence of noise and we are interested in analysing the dynamics when the stable fixed point enters this region of variation of the border. Since the domain relevant for this analysis is close to the border, we consider the border collision normal form of PWS maps which is obtained by linearizing the given piecewise smooth map in that small neighbourhood of the border, and then add the noise term.

Now if the value of the system parameter be such that, both the fixed points x_L and x_R remain outside the interval $(-\delta, \delta)$, then x_L remains the stable fixed point, with $(-\infty, x_R)$ as its basin of attraction. As μ approaches zero, both the fixed points x_L and x_R approach $x = 0$.

Now let the value of a, b in (1) be such that, when μ is sufficiently close to zero, x_L remains outside the interval $(-\delta, \delta)$ but x_R comes inside that interval, i.e., $x_L \notin (-\delta, \delta)$ and $x_R \in (-\delta, \delta)$. This implies

$$x_L < -\delta < x_R < \delta.$$

In that case, starting from any point within the interval $(-\infty, x_R)$, the iterates of (2) ultimately converge to the stable fixed point x_L. Whereas any orbit of (2), starting inside (δ, ∞) diverges to infinity. But what if an orbit starts from any particular point lying inside the interval (x_R, δ)? In case of the deterministic system (1), we know that starting from any point which lies to the right hand side of the unstable fixed point $x = x_R$, the orbit diverges to infinity. Now the question is, will it be the same for the non-deterministic system (2) also or will there be some different scenario? In other words, what will be the future dynamics of the orbit, which starts from any particular point inside (x_R, δ)? Will it converge to the stable fixed point or diverge to infinity?

In this paper, our main aim will be to investigate the above question. We show later that, the resulting orbit may either converge to the stable fixed point x_L or may diverge to infinity and there are non-zero probabilities associated with both these possibilities, i.e., the convergence or divergence of the orbit is not an deterministic event if it starts from a point inside (x_R, δ).

3. Escaping probability of an orbit in minimum iterations

Next we determine the minimum number of iterations needed to escape from the interval (x_R, δ) through the right side of the interval, starting from any arbitrary point $x_0 \in (x_R, \delta)$. In that case starting from x_0, all the iterates should lie to the right hand side of the border till the orbit crosses $x = \delta$. Then the position of the i-th iterate will be

$$x_i = \mu \left(\frac{1 - b^i}{1 - b} \right) + b^i x_0$$