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Rescaling the nonadditivity parameter in Tsallis thermostatistics

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ABSTRACT

The paper introduces nonadditivity parameter transformation group induced by Tsallis entropy. We discuss simple physical applications such as systems in the contact with finite heat bath or systems with temperature fluctuations. With help of the transformation, it is possible to introduce generalized distributive rule in q-deformed algebra. We focus on MaxEnt distributions of Tsallis entropy with rescaled nonadditivity parameter under escort energy constraints. We show that each group element corresponds to one class of q-deformed distributions. Finally, we briefly discuss the application of the transformation to Jizba–Arimitsu hybrid entropy and its connection to Average Hybrid entropy.

1. Introduction

Nonadditive thermodynamics is a concept generalizing ordinary additive Boltzman–Gibbs statistical physics based on Shannon entropy [1]. The first additive generalization of Shannon entropy was discovered by Rényi [2], who showed that the solution of additive Khichin axioms is the one-parametric class of entropies, including Shannon entropy as a special case. The first nonadditive entropy was described by Tsallis [3] and its axiomatic definition was discussed e.g. in [4]. Interestingly, the same entropy functional had been discussed before by Havrda and Charvát in connection with information theory [5]. Since that, there have been introduced many other generalizations of Shanonn entropy [6–9]. Additionally, there were done several successful classifications of generalized entropies taking into account different aspects of generalized statistics [10–14]. On the other hand, Tsallis entropy represents the most popular example of nonadditive entropy with many applications in statistical physics [15–19], abstract algebra [20–22], information theory [23,24] or statistics [25].

In recent years, several attempts on mixing of additivity and nonadditivity effects were studied. To these effects belong crossover between ordinary Gaussian distributions and q-Gaussian distributions [26,27]. We focus on situations, when a system is described by Tsallis entropy, but the strength of nonadditivity can change. This situation can be represented by a simple example of a system in contact with finite heat bath [15,16] with rescaled number of particles in the bath or a system with temperature fluctuations [28,29]. More generally, this corresponds to the situation when we have a system with polynomially growing state space [10,30] and shift its characteristic scaling exponent. Rescaling of nonadditivity parameter brings about some non-trivial consequences which are also discussed in this paper. The rest of the paper is organized as follows: section 2 defines the transformation group of nonadditivity parameter and discusses its main properties. Section 3 discusses simple physical applications of the transform. Section 4 describes applications in q-deformed algebra and presents generalized distributive laws. In section 5 are calculated MaxEnt distributions (obtained from Maximum entropy procedure) corresponding to Tsallis entropy with rescaled nonadditivity parameter under escort energy constraints. Section 6 discusses the connection of the transformation to Jizba–Arimitsu hybrid entropy, which follows q-additivity rule for independent events, similarly to Tsallis entropy. The last section is devoted to conclusions.

2. Rescaling of Tsallis nonadditivity parameter

Nonadditive statistical physics has been first described by Tsallis [3]. He introduced the generalized entropy functional

$$S_q(P) = \frac{\sum_i p_i^q - 1}{1-q} \tag{1}$$

where q plays the role of nonadditivity parameter. For $q = 1$ the entropy becomes additive Shannon entropy. The nonadditivity of Tsallis entropy is for independent events A, B expressed by the axiom

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$$S_q(P_{A\cup B}) = S_q(P_A) + S_q(P_B) + (1-q)S_q(P_A)S_q(P_B) \tag{2}$$

where P_A is the probability distribution corresponding to A, and similarly with B. One of the possible sources of nonadditivity can arise from the fact that a system is in contact with a finite heat bath. In this case, it is possible show that for a bath consisting of N particles the nonadditivity parameter is defined as $q(N) = \frac{N}{N-1}$ [15,16], and the state space grows polynomially. The polynomial growth of states is typical for systems described by Tsallis entropy, because the entropy is extensive for these systems [10]. This is typical for systems where a fraction of states is frozen [30]. The important question is what happens with the system when we change the strength of non-additive interaction. This means that we rescale the nonadditivity term in Eq. (11), i.e. we replace $(1-q)$ by $(q-1)/\alpha$ for some $\alpha > 0$. In this case, we get a new nonadditivity parameter q_α, which is defined by the relation

$$(q_\alpha - 1) = \frac{q-1}{\alpha} \qquad \Rightarrow \qquad q_\alpha = \frac{q+\alpha-1}{\alpha}. \tag{3}$$

Interestingly, this class of nonadditivity parameter transformations conforms a one-dimensional group. It is straightforward to show the main properties of the group, which are

- composition rule: $(q_\alpha)_\beta = (q_\beta)_\alpha = q_{\alpha\cdot\beta}$
- associativity: $(q_\alpha)_{\beta\cdot\gamma} = (q_{\alpha\cdot\beta})_\gamma = q_{\alpha\cdot\beta\cdot\gamma}$
- neutral element: $q_1 = q$
- transformation invariant: $1_\alpha \equiv 1$.

Naturally, it is possible to think about extension of the transform beyond the region $\alpha > 0$. Actually, the region $\alpha > 0$ rescales the distance of q from 1, but does not change the sign of $(1-q)$. It means that the transform keeps $q_\alpha > 1$ for $q > 1$ and vice versa. Because of multiplicative properties of the transform, the most important is the extension to $\alpha = -1$. With this, we get

$$q_{-1} = \frac{q-1-1}{-1} = 2-q \tag{4}$$

which is the well-known additive duality of Tsallis entropy. Unfortunately, this transformation can lead to negative values of q_{-1}, which is usually unwanted because Tsallis entropy for negative values of q does not fulfill Kolmogorov axioms defined by Abe [4]. This can be overcome by assuming only $q \in [0,2]$.

It is also possible to obtain the multiplicative duality, when we allow q-dependent transformation $\alpha(q)$. In this case, we simply choose $\alpha(q) = -q$, which results into

$$q_{-q} = \frac{q-q-1}{-q} = 1/q. \tag{5}$$

Both additive and multiplicative dualities have been recently discussed e.g. in Ref. [31].

3. Applications of Tsallis parameter transformation in thermostatistics

In order to understand the physical interpretation of the transformation $S_q \to S_{q_\alpha}$, let us focus on the case of finite heat bath. For this system is

$$q(N)_\alpha = \frac{q(N)+\alpha-1}{\alpha} = \frac{N+(\alpha-1)(N-1)}{\alpha(N-1)} = \frac{\alpha(N-1)+1}{\alpha(N-1)}. \tag{6}$$

Therefore, parameter q_α corresponds to a system in contact with a finite heat bath consisting of $N_\alpha = \alpha(N-1)+1$ particles, from which we get that

$$(N_\alpha - 1) = \alpha(N-1). \tag{7}$$

Thus, transformation $q \to q_\alpha$ corresponds to rescaling the number of particles in the bath. Generally, the transform describes the shift between classes of Tsallis q-additivity. For the case of finite heat bath, we always have $q(N)_\alpha > 1$ for $\alpha > 0$.

Rescaling the number of particles in the finite heat bath is one of the physical applications of the nonadditivity parameter transformation. On the other hand, it is possible to find a nice physical interpretation of the transformation for systems with temperature fluctuations. Such systems have been investigated by Beck in Ref. [28] followed by several other authors. In a general system in contact with a heat bath with temperature fluctuations it is possible to express the non-additivity parameter q as [29]

$$q = 1 - \frac{1}{C} + \frac{\Delta\beta^2}{\langle\beta\rangle^2} \tag{8}$$

where C is the heat capacity of the reservoir and $\frac{\Delta\beta^2}{\langle\beta\rangle^2}$ is the relative temperature fluctuation. Let us note for the case of finite heat reservoir discussed in the previous section, the heat capacity is negative, as discussed e.g. in Ref. [32]. On the other hand, for positive heat capacity, it is possible to reach the region $q < 1$ and $q = 1$ determines $\frac{1}{\sqrt{C}} = \frac{\Delta\beta}{\langle\beta\rangle}$. For systems, with large fluctuations, i.e. $\frac{\Delta\beta^2}{\langle\beta\rangle^2} \gg \frac{1}{C}$, we can neglect $1/C$. Then, for the system with rescaled nonadditivity parameter q_α, we have

$$\frac{\Delta\beta_\alpha^2}{\langle\beta_\alpha\rangle^2} = q_\alpha - 1 = \frac{q-1}{\alpha} = \frac{1}{\alpha}\frac{\Delta\beta^2}{\langle\beta\rangle^2}. \tag{9}$$

Thus, rescaling the nonadditivity parameter also rescales the relative fluctuations in the system.

Finally, a nice application of the nonadditivity parameter transformation is the *quasi-additivity* rule for Tsallis entropy for q close to one [28]. In this case, it is possible to make the expansion of $\sum_i = p_i^q$ as

$$\sum_i p_i^q = \sum_i p_i e^{(q-1)\log p_i} = 1 + (q-1)\sum_i p_i \log p_i + \frac{(q-1)^2}{2}\sum_i p_i (\log p_i)^2 + \dots \tag{10}$$

In this approximation it is possible to find a quasi-additivity rule for Tsallis entropy, which can be expressed as

$$S_q(P_A) + S_q(P_B) = S_{q_\alpha}(P_{A\cup B}) \tag{11}$$

with appropriate α. For $A = B$, the left-hand side is equal to

$$2S_q(P_A) = \frac{2}{q-1}\left(1 - \sum_i p_i^q\right) = -2\sum_i p_i \log p_i - (q-1)\sum_i p_i (\log p_i)^2 + \dots \tag{12}$$

while the right-hand side can be expressed as

$$S_{q_\alpha}(P_{A^2}) = \frac{1}{q_\alpha - 1}(1 - \sum_{ij} p_{ij}^{q_\alpha}) = \frac{1}{q_\alpha - 1}(1 - (\sum_i p_i^{q_\alpha})^2) = -2\sum_i p_i \log p_i - (q_\alpha - 1)\left[(\sum_i p_i \log p_i)^2 + \sum_i p_i (\log p_i)^2\right] + \dots \tag{13}$$