Physics Letters A 381 (2017) 2598–2607

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

A tight Cramér–Rao bound for joint parameter estimation with a pure two-mode squeezed probe

Mark Bradshaw *, Syed M. Assad, Ping Koy Lam

Centre for Quantum Computation and Communication Technology, Department of Quantum Science, Research School of Physics and Engineering, Australian National University, Canberra, ACT 2601, Australia

ARTICLE INFO

Article history:
Received 21 February 2017
Received in revised form 14 June 2017
Accepted 14 June 2017
Available online 21 June 2017
Communicated by P.R. Holland

Keywords:
Quantum physics
Quantum information
Quantum optics
Parameter estimation
Cramér–Rao bound

ABSTRACT

We calculate the Holevo Cramér–Rao bound for estimation of the displacement experienced by one mode of an two-mode squeezed vacuum state with squeezing r and find that it is equal to $4\exp(-2r)$. This equals the sum of the mean squared error obtained from a dual homodyne measurement, indicating that the bound is tight and that the dual homodyne measurement is optimal.

1. Introduction

In quantum mechanics, there is a limit to the precision to which we can simultaneously measure two observables that do not commute. If the observables are complementary variables, such as position and momentum, this limit is described by the Heisenberg uncertainty principle.

In continuous variable quantum optics, the bosonic quadrature operators Q and P are another pair of complementary variables, and obey the canonical commutation relation $[Q, P] = 2i$, where throughout the paper we use units where $\hbar = 2$. The bosonic field is also described by the annihilation operator a and creation operator $a^\dagger$, which are related to the quadrature operators by $Q = a + a^\dagger$ and $P = i(a^\dagger - a)$. The displacement operator is given by

$$D(\alpha) = \exp\left(\alpha a^\dagger - \alpha^* a\right), \tag{1}$$

where $\alpha = (q + ip)/2$ is the complex amplitude.

In this paper we address this question: Given the probe state ρ_0 which undergoes a displacement of $D(\alpha)$ resulting in $\rho_\theta = D(\alpha)\rho_0 D^\dagger(\alpha)$, how well can we estimate the two parameters $\theta_1 := q = 2\,\mathrm{Re}(\alpha)$ and $\theta_2 := p = 2\,\mathrm{Im}(\alpha)$? Our figure of merit is sum of the mean square error (MSE), $\mathcal{V} := E[(\hat{\theta}_1 - \theta_1)^2] + E[(\hat{\theta}_2 - \theta_2)^2]$ where E is the expectation value, and $\hat{\theta}_1$ and $\hat{\theta}_2$ are the estimates of θ_1 and θ_2 respectively. We aim to find a lower bound to $\mathcal{V}$. These bounds are called Cramér–Rao bounds (CR bounds) and will only depend on the state ρ_θ and independent of the measurement performed on it. We calculate the CR bound based on the work of Holevo [1,2], which we call Holevo CR bound. First, we calculate the Holevo CR bound when the probe state ρ_0 is a single mode squeezed state, for which tight bounds are already known. Next, we calculate the Holevo CR bound when the probe is a two mode squeezed state. We find that it is superior to bounds calculated by previous authors [3,4]; and that the bound can be reached by a simple measurement.

Our paper is divided up into sections as follows. In section 2, we briefly describe the Gaussian quantum optics used in our results. In section 3, we summarize parameter estimation theory including CR bounds. In section 4, we summarize the bounds found by other authors and the MSE from a dual homodyne measurement. In section 5, we calculate the Holevo CR bound for one- and two-mode squeezed probes and discuss our results.

* Corresponding author.
E-mail address: mark.bradshaw@anu.edu.au (M. Bradshaw).

http://dx.doi.org/10.1016/j.physleta.2017.06.024
0375-9601/

2. Gaussian quantum optics

Consider a state consisting of m bosonic modes. Let the annihilation and creation operators of the kth mode be a_k and $a_k^\dagger$, respectively, and the quadrature operators be Q_k and P_k. Define a vector Z to contain all the quadrature operators:

$$\vec{Z} = (Q_1, P_1, ..., Q_m, P_m) \tag{2}$$

The mean of the quadrature operators of a state ρ, otherwise known as the displacement vector of the state, is given by

$$M = \left[\langle Z_j\rangle\right]_j , \tag{3}$$

where $\langle A\rangle = \mathrm{tr}(\rho A)$ is the expectation value of operator A, and $\mathrm{tr}(\cdot)$ denotes trace of a density matrix. Define the covariance matrix V, which contains the variances of the quadrature operators, by

$$V = \left[\frac{1}{2}\langle Z_j Z_k - Z_k Z_j\rangle - \langle Z_j\rangle\langle Z_k\rangle\right]_{jk} . \tag{4}$$

A thermal state is given by

$$\rho_{\mathrm{th}}(N) = \frac{1}{N+1}\sum_{n=0}^{\infty}\left(\frac{N}{1+N}\right)^n |n\rangle\langle n| \tag{5}$$

where $|n\rangle$ are the Fock states, and N is the mean number of photons in the bosonic mode. A thermal state has a zero displacement vector and a covariance matrix of $V_{\mathrm{th}} = (2N+1)I_2$ where I_2 is the 2×2 identity matrix.

The single-mode squeezing operator is given by

$$S(r) = \exp\left(\frac{1}{2}(ra^2 - ra^{\dagger 2})\right), \tag{6}$$

where r is the squeezing parameter. When acting on the vacuum state, this gives the squeezed vacuum state $|S(r)\rangle = S(r)|0\rangle$. The squeezed vacuum state has a zero displacement vector and a covariance matrix of

$$V_{\mathrm{sq}} = \begin{pmatrix} \mathrm{e}^{-2r} & 0 \\ 0 & \mathrm{e}^{2r} \end{pmatrix} . \tag{7}$$

The two-mode squeezing operator is given by

$$S_2(r) = \exp\left(ra_1a_2 - ra_1^\dagger a_2^\dagger\right). \tag{8}$$

When acting on two vacuum states it gives the two-mode squeezed vacuum state, also known as the Einstein–Podolski–Rosen (EPR) state. The two-mode squeezed vacuum state has zero displacement vector and covariance matrix of

$$V_{\mathrm{EPR}} = \begin{pmatrix} \cosh(2r) & 0 & \sinh(2r) & 0 \\ 0 & \cosh(2r) & 0 & -\sinh(2r) \\ \sinh(2r) & 0 & \cosh(2r) & 0 \\ 0 & -\sinh(2r) & 0 & \cosh(2r) \end{pmatrix} . \tag{9}$$

A beam splitter is used to mix two modes. It is described by the unitary transformation

$$B(\phi) = \exp\left(\phi(a_1^\dagger a_2 - a_1 a_2^\dagger)\right) \tag{10}$$

where $\tau = \cos^2\phi$ is the transmissivity of the beam splitter.

3. Parameter estimation theory

Let ρ_θ be a family of states parametrized by d parameters $\theta = (\theta_1, \theta_2, ..., \theta_d)$. The goal of parameter estimation is to estimate the value of θ based on the outcome of a measurement on ρ_θ. In quantum mechanics, a measurement is described by a positive operator-valued measure (POVM) $\Pi = \{\Pi_x\}$. Each measurement outcome x has a corresponding non-negative hermitian operator Π_x associated with it, where the probability of measuring x on a state ρ_θ is $p_\theta(x) = \mathrm{tr}(\Pi_x\rho_\theta)$, and the POVM elements sum to Identity: $\sum_x \Pi_x = I$. We then need an estimator $\hat{\theta}(x)$, which maps the observed outcome x to an estimate for θ. An estimator is called locally unbiased at θ if $E[\hat{\theta}(x)] = \theta$ at the point θ. An estimator is called unbiased if and only if it is locally unbiased at every θ.

The MSE matrix $V_\theta[\hat{\theta}]$ of the estimator $\hat{\theta}$ is given by

$$V_\theta[\hat{\theta}] = \left[\sum_x p_\theta(x)(\hat{\theta}_j(x) - \theta_j)(\hat{\theta}_k(x) - \theta_k)\right]_{jk} . \tag{11}$$

The sum of the MSE $\mathcal{V}$ is the trace of the MSE matrix:

$$\mathcal{V} = \mathrm{Tr}\left\{V_\theta[\hat{\theta}]\right\}. \tag{12}$$

Here, $\mathrm{Tr}\{\cdot\}$ denotes the trace of an estimator matrix. The Cramér–Rao bound provides a lower bound to the MSE matrix for a classical probability distribution $p_\theta(x)$: