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Extended class of phenomenological universalities

Marcin Molski

Faculty of Chemistry, Theoretical Chemistry Department, Adam Mickiewicz University, Umultowska 89b, 61-614 Poznań, Poland

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ABSTRACT

The phenomenological universalities (PU) are extended to include quantum oscillatory phenomena, coherence and supersymmetry. It will be proved that this approach generates minimum uncertainty coherent states of time-dependent oscillators, which in the dissociation (classical) limit reduce to the functions describing growth (regression) of the systems evolving over time. The PU formalism can be applied also to construct the coherent states of space-dependent oscillators, which in the dissociation limit produce cumulative distribution functions widely used in probability theory and statistics. A combination of the PU and supersymmetry provides a convenient tool for generating analytical solutions of the Fokker–Planck equation with the drift term related to the different forms of potential energy function. The results obtained reveal existence of a new class of macroscopic quantum (or quasi-quantum) phenomena, which may play a vital role in coherent formation of the specific growth patterns in complex systems.

1. Introduction

The concept of PU introduced by Castorina, Delsanto, and Guiot (CDG) [1,2] concerns ontologically different systems, in which miscellaneous emerging patterns are described by the same mathematical formalism. Universality classes are useful for their applicative relevance and facilitate the cross fertilization among various fields of research, including physics, chemistry, biology, ecology, engineering, economics and social sciences [1–18]. This strategy is extremely important, especially for the export of ideas, models and methods developed in one discipline to another and vice versa. The PU approach is also a useful tool for investigation of the complex systems whose evolution is governed by nonlinear processes. Hence, this methodology can be employed [1] to obtain different functions of growth widely applied in actuarial mathematics, biology and medicine. In this work the research area is extended to include in the CDG scheme the quantum oscillatory phenomena, coherence and supersymmetry. In particular it will be proved that the CDG formalism is a hidden form of supersymmetry, which can be employed not only to produce macroscopic growth functions but also to construct quantum coherent states of the time- and space-dependent Morse [20] and Wei [21] oscillators. In the dissociation (classical) limit they reduce to the well-know Gompertz [22] and West–Brown–Enquist (WBE)-type [23] functions (e.g. logistic, exponential, Richards, von Bertalanffy) describing sigmoidal (S-shaped) temporal evolution or spatial distribution of subelements of complex systems. We shall also be concerned with a generalization of the CDG approach to include regression states which has not been consider in the original formulation of the CDG theory.

2. Theory

According to the CDG theory, various degrees of nonlinearity appearing in the complex systems can be described and classified using the set of nonlinear equations [1]

$$\frac{d\psi(q)}{dq} - x(q)\psi(q) = 0, \qquad \frac{dx(q)}{dq} + \Phi(x) = 0. \tag{1}$$

Here, $q = u_t t$ denotes dimensionless temporal variable, u_t is a scaling factor, whereas $\Phi(x)$ stands for a generating function, which expanded into a series of x-variable (it slightly differs from the original CDG formulae) [1]

$$\Phi(x) = c_1(x + c_0/c_1) + c_2(x + c_0/c_1)^2 + \dots \tag{2}$$

produces different functions of growth $\psi(q)$ for a variety of patterns emerging in the systems under consideration. To obtain their explicit forms a combination of Eqs. (1) is integrated generating the growth functions [1]

E-mail address: mamolski@amu.edu.pl.

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$$\psi(q) = \exp\left[-\int_x \frac{xdx}{\Phi(x)} + C\right] = \exp\left[\int_q x(q)dq + C\right] \quad (3)$$

for different powers $n = 1, 2, ...$ of the truncated series (2). The integration constant C can be calculated from a boundary condition $x(q = 0)$. For example for $x(0) = 1$, $c_0 = 0$, $c_1 = 1$, $\Phi(x) = x$ one gets the Gompertz function [22], whereas for $\Phi(x) = x + c_2 x_2$ the allometric WBE-type function [23] can be derived [1]

$$\psi(q)_G = \exp\left[(1 - \exp(-q)\right],$$
$$\psi(q)_W = \exp\left[1 + c_2 - c_2 \exp(-q)\right]^{1/c_2}. \quad (4)$$

Employing this approach the PU can be classified [1] as $U1$ $(n = 1)$, $U2$ $(n = 2)$ etc. with respect to the different levels of nonlinearity utilized by the complex systems during formation of the specific growth patterns. In subsequent works [9,10] the PU concept has been extended to include parameter c_1 and $c_0 = \Phi(x) = 0$. In the latter case, the solutions of Eqs. (1) and (3) take the form $x(q) = x(0)$, $\psi(q) = \exp[x(0)q]$, which for $q = u_t t$ represents $U0$ class of PU describing self-catalytic processes [10]. The solutions of Eqs. (1) and (3) obtained in the original CDG scheme take the form monotonic growth curves, with no allowance for oscillations, which are ubiquitous dynamic feature of the complex systems observed in nature. Oscillations are usually the results of the mutual interferences between a growing system and its surrounding, or between several competing processes appearing in the same system. To describe such phenomena Barberis et al. [11] employed a complex function, whose real and imaginary parts represent two phenotypic traits of the same organism. As the result a generalization of the Gompertz and WBE growth models has been obtained. The PU complex field formalism has been applied also by Delsanto et al. [9] in analyzing the evolution of the system depending on two variables driven by the set of coupled nonlinear differential equations (1). They reproduced main oscillatory features of the time-evolution curves belonging to the complex counterparts of $U1$ and $U2$ classes of PU. In another model proposed by Barberis and co-workers [12,13] interactive growth phenomena in biological and ecological systems are described using vector formulation of PU in the real space. In this way the joint growth of two or more interacting organisms as well as mutual influences that operate through environment modifications can be characterized without *ad hoc* formulated assumptions on the nature of the interactions. The vector universalities model has been also applied to describe the cancer growth viewed as the result of the competition between two or more cancer cell populations [14]. Recently, the classical oscillations have been also considered [17] in the CDG scheme generalized by Molski [19] employing only real variable. In this letter it will be proved that the PU classification scheme embraces not only classical but also quantum oscillatory phenomena, whose complete characteristics can be determined without the use of complex formalism. On the other hand, the proposed supersymmetric interpretation of the CDG theory is based on the two component (vector) functions build up with the growth and regression terms, hence we find here a some analogy to the Barberis and co-workers formalism [12–14]. However, it should be pointed out that the growth-regression states represent independent and uncorrelated in time processes, so the continuous transition of the growth phase to the regression state and *vice versa* is a genuine property of the systems under consideration.

3. Results

The CDG approach can be easily extended to include the space-dependent phenomena using spatial variable $q = u_r r$ in which u_r is a scaling factor. In this way one may generate in the CDG scheme the space-dependent sigmoidal Gompertz and WBE-type functions widely applied in a range of fields including e.g. probability theory and statistics where they are used to describe cumulative distribution of entities characterized by different spatial sizes [24]. In view of this the CDG formalism can describe not only temporal evolution of the complex systems represented by $\psi(t)$ but also the spatial distribution $\psi(r)$ of their subcomponents. In particular, the spatial version of the Gompertz function (4) has been found as a powerful descriptive tool for neuroscience where can be used as cumulative distribution curve correctly describing diameters of fibers in the olfactory nerves [24].

3.1. Regression

A detailed analysis of the CDG approach reveals that it does not take into account a very important phenomenon of regression (decay) appearing in biological, medical, demographic and economic systems. Such an effect appears, for example, under chemotherapeutic treatment of tumors subjected to cycle specific (or nonspecific) drugs causing regression of cancer whose growth (decay) is described by the Gompertz function [25]. Recent investigations in the field of neurology revealed also that the temporal Gompertz function of regression can be employed to describe time course of synaptic current or change the membrane conductance during voltage clamp of squid axon [24]. To include the regression phenomenon in the CDG scheme, Eqs. (1) and (3) should be modified to the form

$$\frac{d\psi(q)^\dagger}{dq} + x(q)\psi(q)^\dagger = 0, \quad (5)$$

$$\psi(q)^\dagger = \exp\left[\int_x \frac{xdx}{\Phi(x)} + C\right] = \exp\left[-\int_q x(q)dq + C\right], \quad (6)$$

which for $\Phi(x) = x$ and $\Phi(x) = x + c_2 x^2$ produce Gompertz and WBE-type functions of regression [24,25]

$$\psi(q)^\dagger_G = \exp\left[-(1 - \exp(-q)\right],$$
$$\psi(q)^\dagger_W = \exp\left[1 + c_2 - c_2 \exp(-q)\right]^{-1/c_2}. \quad (7)$$

It is easy to demonstrate that for $q \to \infty$ the functions $\psi(q)^\dagger_G \to \exp(-1)$, $\psi(q)^\dagger_W \to (1 + c_2)^{-1/c_2}$ diminish with q, hence they describe the regression states of the system under consideration. The regression states have been generated also in [18] by a proper choice of the parameters defining the growth functions or by applying the involuted Gompertz function derived by Molski [19]. In contrast to this approach Eq. (5) can be applied to derive regression states associated with all types of growth functions created in the CDG scheme independently of parameters defining them.

3.2. Supersymmetry

Analysis of Eqs. (1) and (5) reveals that they can be interpreted in the framework of time-dependent $(q = u_t t)$ [26] or space-dependent $(q = u_r r)$ [27] quantum supersymmetry (SUSYQM), used among others to construct coherent states of quantum oscillators and to obtain exact solutions of the Schrödinger equation for vibrating harmonic and anharmonic systems. In view of this, it is tempting to apply the CDG methodology to generate the coherent states of time- and space-dependent oscillators and compare them with those obtained using algebraic procedure [28,29]. To prove that mathematical formalism of PU is a hidden form of supersymmetry, lets differentiate growth equation (1) once with respect to q-coordinate and then rearrange the derived formulae to obtain the second order differential equation in a standard eigenvalue form