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The geometro-hydrodynamical representation of the torsion field ☆

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A B S T R A C T

We construct the geometro-hydrodynamical formalism for a spinning particle based on the six-dimensional manifold of autoparallelism geometry which is represented as a vector bundle with a base formed by the manifold of the translational coordinates and a fiber specified at each point by the field of an orthogonal coordinate frame underlying the classical spin. We show that the geometry of oriented points leads to the existence of torsion field with the source – the classical spin. We expand the geometro-hydrodynamical representation of Pauli field developed by Takabayasi and Vigier. We show that the external torsion field has a force effect on the velocity and spin fields via the spin-vorticity, which is characteristic of the space structure with the inhomogene triad field. The possible experimental effects of torsion field are discussed.

1. Introduction

The main concept of quantum-mechanical theory of a single non-relativistic particle with spin is a spinor wave field. The construction of a realistic physical model of a quantum matter on the way of representing visually the information encoded in a spinor was undertaken in Refs. [1–5]. The new geometrical representation of a two-component spinor with an orthogonal set of coordinate axes was developed by Kramer [1]. The representation of Pauli field as an assembly of very small rotating particles continuously distributed in space was later evolved in Ref. [2] using Kramers' idea. The hydrodynamical consideration describes the Pauli field as a fluid of small particles having intrinsic angular momentum or spin. The geometrical representation introduces a triad structure underlying the classical spin [3] where each fluid element is represented by this triad and spin vector is fixed to the third axis of the triad. Based on this, the spinor is interpreted as defining a state of rotation [6]. This approach leads to a concept of the spinor ψ in terms of Euler angles which determine the orientation of the triad with respect to the fixed set of Cartesian axes [3]. The realistic geometro-hydrodynamical interpretation of the quantum matter is based on the postulate that embedded in the wave is a particle of mass m, which transfers energy and momentum. The evolution of the spinor field, which must represent a new physical field, implies the rotation of the triad and, as a result, the movement of the center of mass of the particle. The transformation and orientation of the ensemble of spinning particles were described in the field variables (basic quantities ρ, $\mathbf{s}$, $\mathbf{v}$) on the basis of the conservation equation for the spatial distribution of this ensemble [4,6] $\rho(\mathbf{r},t)=\psi^{+}\psi$

$$\partial_t\rho + \nabla(\rho\mathbf{v}) = 0, \tag{1}$$

of the velocity field $\mathbf{v}(\mathbf{r},t)$ evolution equation or Euler equation of flow [6]

$$m(\partial_t + \mathbf{v}\nabla)\mathbf{v} = e\mathbf{E} + \frac{e}{c}\mathbf{v}\times\mathbf{B} - \nabla(\Pi + \Pi_{in}) + \frac{e}{mc}s_k\nabla(B^k + B^k_{in}), \tag{2}$$

and of the spin field $\mathbf{s}(\mathbf{r},t)$ equation of motion [6]

$$(\partial_t + \mathbf{v}\nabla)\mathbf{s} = \frac{2\mu}{\hbar}\mathbf{s}\times(\mathbf{B}+\mathbf{B}_{in}). \tag{3}$$

In this hydrodynamical representation the motion of non-relativistic quantum-mechanical spinning particle is carried out in such a way that each element of the fluid evolves like a classic spinning particle under the action of external forces and of the internal potential $\Pi_{in} = -|\nabla\mathbf{s}|^2/2m$ and the internal magnetic field $\mathbf{B}_{in} = c/e(\triangle\mathbf{s} + \partial_k\rho\cdot\partial_k\mathbf{s}/\rho)$.

2. Geometrical representation

We use the geometro-hydrodynamical scheme that had been evolved in the works of Takabayasi, Bohm, Vigier and Holland [1–6] and consider the Pauli field as a fluid of spinning elements. In this section we introduce the internal configurational variables that are attributed to each fluid element. To take into account the rotational degrees of freedom in non-relativistic case we have to

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use the three-dimensional manifold of oriented points. The simplest generalization of the three-dimensional Euclidian geometry to the case of the manifold with all its points oriented is the geometry of absolute parallelism (autoparallelism $A_3(3)$) constructed on the six-dimensional manifold which is represented as a vector bundle with a base formed by the manifold of the translational coordinates and a fiber specified at each point by the field of an orthogonal coordinate frame or triad $\mathbf{e}_{(a)}(\mathbf{x},t)$, which vary from point to point or depending on $\mathbf{x}$, where a is the number of the reference vector [7–9]. The autoparallelism was used by Albert Einstein in his attempts to construct a classical unified field theory [10], which was based on the Riemann metric and autoparallelism. He had developed this geometry in the n-dimensional differentiable manifold. The geometry with absolute parallelism was first considered in the works of Weitzenbock and Vitali [11,12]. Weitzenbock suggested the existence of the n-dimensional manifold with coordinates of Riemannian spaces with a zero curvature tensor.

The orientation of the triad with respect to the fixed set of space axes is defined by the space-dependent set of Euler angles $\phi(\mathbf{x},t)$, $\vartheta(\mathbf{x},t)$, $\chi(\mathbf{x},t)$. The system of vectors $\mathbf{e}_{(a)}$, which are defined at each point of space, is orthonormal, determines a state of rotation $e_i^{(a)}\cdot e^j_{(a)}=\delta_i^j$ and exists as basic vectors defined and translatable in the absolute sense to any point of space in any direction. At every world point, the orientation of the body frame can be specified referring to the triad the origin of which coincides with the center of mass of the particle at that moment [13]. Using the approach developed in [5,6], we introduce the matrix transformation a_{ij} denoting the components of triad $\mathbf{e}_i^{(a)}$ with respect to the space axes in standard orientation $\mathbf{e}_i=a_{ij}\mathbf{c}^j$. The tensor of angular velocity of the rotation of the reference frame ω_k^j can be determined by the relation

$$\omega_{jk}=\mathbf{e}_k\cdot\frac{d\mathbf{e}_j}{dt},\qquad \omega_{jk}=-\omega_{kj}. \tag{4}$$

Now it is natural to introduce the main properties of absolute parallelism geometry (autoparallelism). The main consequence of this geometry that the torsion is a characteristic of the space. The concept of connection of geometry can be written as $\Delta^i_{jk}=\Gamma^i_{jk}+\gamma^i_{jk}$, where the Christoffel symbols $\Gamma^i_{jk}=1/2g^{im}(\partial_k g_{jm}+\partial_j g_{km}-\partial_m g_{jk})$, which depend on the metric tensor $g_{ij}=e_i^{(a)}e_j^{(a)}$ and γ^i_{jk} is the Ricci rotation coefficients or torsion. The Christoffel symbols are the particular case of the affine connection and in the flat space must vanish. But in the case of the flat space with torsion the geometry is not completely described by the metric only but the independent characteristic – torsion. The general form of the torsion tensor [8,9] and [14] in the coordinate indices is

$$\gamma^i_{jk}=-\Omega^i_{jk}+g^{im}(g_{jn}\Omega^n_{mk}+g_{kn}\Omega^n_{mj}), \tag{5}$$

where Ω^n_{mk} has the form of

$$\Omega^i_{jk}=\frac{1}{2}e^i_{(a)}\left(\partial_j e_k^{(a)}-\partial_k e_j^{(a)}\right), \tag{6}$$

and can be characterized as the object of anholonomity [7,8,14]. In the case of the flat affine space with torsion, where the quantities γ^i_{jk} represent the local spin connection of space and are referred to as the Ricci rotation coefficients for the basis $e_i^{(a)}$. Torsion is an independent characteristic of the space–time and in the anholonomic coordinates the Ricci rotation coefficients are transformed as follows $\gamma^{(a)}_{(b)k}\equiv e_i^{(a)}\gamma^i_{jk}e^j_{(b)}$. A space of events has two metrics – the Riemann flat metric and the three-dimensional Killing–Cartan metric $d\nu^2=d\chi_{ij}d\chi^{ij}$, where the infinitesimal increments can be given by the vector $d\chi^i=\omega^i dt$.

The parallel displacement of the triad relative to the connection Δ^i_{jk} equals zero identically $\partial_k e_j^{(a)}-\Delta^i_{jk}e_i^{(a)}=0$. From this definition the connection can be defined as $\Delta^i_{jk}=e^i_{(a)}\partial_k e_j^{(a)}$ and the Ricci rotation coefficients proportional to the covariant derivative with respect to the Christoffel symbols

$$\gamma^i_{jk}=e^i_{(a)}\nabla_k e_j^{(a)},$$

as a result the angular velocity of the triad must have the geometrical form

$$\omega^i_j=\gamma^i_{jk}\frac{dx^k}{dt}. \tag{7}$$

The angular velocity 3-vector ω^i can be found from the relations defining rigidly rotating Cartesian coordinates and easy to derive that the vector of the triad, which provide a covariant specification of a state of rotation, in the flat space satisfy the equation

$$\frac{de_j^{(a)}}{dt}+\gamma^i_{jk}\frac{dx^k}{dt}e_i^{(a)}=0, \tag{8}$$

which is responsible for the temporal dynamics of the triad vectors. Note that even in zero external magnetic field, when the particle has no magnetic momentum, the vectors of the triad will precess due to the action of the torsion torque, which will be derived below. We resort to the description in the set of variables ρ, $\mathbf{e}_{(a)}$ and $\mathbf{v}$. The main dynamical property of the triad is that its angular momentum of rotation or spin vector is fixed to the third axis of triad and has the magnitude $\hbar/2$

$$\mathbf{s}=\frac{\hbar}{2}\mathbf{\Sigma},\qquad \text{where}\qquad \mathbf{\Sigma}=\mathbf{e}^{(3)}=\mathbf{e}^{(1)}\times\mathbf{e}^{(2)}. \tag{9}$$

In the terms of Euler angles, the polarization – is the proper internal degrees of freedom, can be represented as $\Sigma_1=\sin\vartheta\cos\phi$, $\Sigma_2=\sin\vartheta\sin\phi$ and $\Sigma_3=\cos\vartheta$, where angles ϑ, ϕ denote the polar angles of polarization, but χ characterizes the rotational orientation of the orthogonal vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ axes on the plane normal to polarization vector.

2.1. The Pauli equation

Let's move on to the relativistic task and consider a four-dimensional manifold such that at each point of space the tetrad is specified. One introduces the field of the tetrad e^a_β, where Greek indices are the coordinate indices and refer to the generic inertial frame. Tetrad defines the fundamental metric tensor $g_{\alpha\beta}=e^a_\alpha e^b_\beta\eta_{ab}$, where η_{ab} – is the Minkowski metric. We also discount, that the covariant derivative in nonholonomic coordinates has the form $\nabla_a F^b=\partial_a F^b+\Delta^b_{ac}F^c$. The wave equation of the spinning particle exhibit the properties that relates to an general requirements space–time symmetry. The spinor wave function Ψ for a spin-1/2 particle of rest mass m and charge e in space–time with torsion obeys the Dirac equation in Cartesian coordinates

$$\gamma^\mu D_\mu\Psi+\frac{imc}{\hbar}\Psi=0, \tag{10}$$

where we obtained the sought equation by replacing the operator of 4-momentum and introduce the covariant derivative of a spinor. The affine connection $\Delta^\alpha_{\beta\mu}$ different from Christoffel symbol means that the geometry is not completely described by the metric, but has another, absolutely independent characteristic – the tensor [20]. The covariant derivative can be formed from the sum of partial derivative and the additional term – torsion

$$D_\mu=\partial_\mu-\frac{q}{c}A_\mu-\gamma_\mu, \tag{11}$$