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Relativistic dipole interaction and the topological nature for induced HMW and AC phases

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ABSTRACT

In this work we give, for the first time, the full relativistic Lagrangian density describing the motion of induced electric dipoles in the electric fields which induce the dipole, and the magnetic fields which generate the HMW topological phase. We then use this relativistic Lagrangian density to derive the complete set of conditions for producing topological phases with induced dipoles. We also give the relativistic Lagrangian density describing the motion of induced magnetic dipoles in the magnetic fields which induce the dipole, and the electric fields which generate the AC topological phase, and derive the conditions for this AC phase to be topological. These conditions have been incompletely discussed in previous studies. We note that, in both the AC and HMW cases, the topological phases are generated by the cross product of electric and magnetic fields in the form $\boldsymbol{B} \times \boldsymbol{E}$ which reinforces the dual nature of these two topological phases.

The topological phase is one of the most important aspects of quantum mechanics which distinguish quantum from classical mechanics. Some of the key examples come from quantum mechanical electromagnetic interactions. There are three quantum mechanical electromagnetic topological phases, the Aharonov–Bohm (AB) phase [1], the Aharonov–Casher (AC) Phase [2] and the He–McKellar–Wilkens (HMW) phase [3–5], which have been experimentally verified. These phases reside in the phase factor of the wave function. A common feature of the topological nature of these phases is that when a particle, carrying a certain "charge" which induces interactions with external electric and/or magnetic fields, travels through regions where, in classical sense, no forces act on the particle, but when it encircles a closed path which contains certain field configurations (FC) which the particle does not enter, the wave function develops a non-trivial phase independent of the particular path travelled as long as it encloses the given FC [6]. In the AB case, the charge is the electric charge and the FC is magnetic flux. In AC and HWM cases, the charges are magnetic dipole and electric dipole and the FCs are field configurations in which the vector product of the magnetic or electric dipole and the electric or magnetic field has a non-vanishing curl. These non-trivial phases cause interference effects which have been observed experimentally, the topological AB phase by Chambers [7] and by Tonomura et al. [8], the topological AC phase by Cimmino et al. [9] and the Toulouse group [10], and the topological HMW effect by the Toulouse group [11], and the Tokyo Atom Interferometry Group [12]. For the HMW effect, because no atom has an observable electric dipole moment, it is necessary to induce the electric dipole moment by applying an electric field [4]. Using an induced electric dipole moving in a magnetic field also means that the phase can be topological without needing a magnetic monopole source of the magnetic field [13]. Recent reviews of electromagnetic topological phases have been given by BMcK in Ref. [14] and Ref. [15].

A crucial ingredient in identifying a topological phase is to analyse the relevant Hamiltonian governing the motion of a particle to see if there are certain configurations so that there is a term (terms) H_{top} which exhibits the feature that they do not exert force on the particle and therefore can be transformed into the wave function producing a phase factor $e^{-iH_{top}\Delta t}$. When the particle is travelling a closed path taking a time T, one integrates the phase factor $\int_0^T H_{top} dt$. One can translate this into an integral along the path, $\oint_C \boldsymbol{T} \cdot \boldsymbol{dr}$. One then checks to see if this integral has a value independent of the path travelled. In order to have a topological phase certain conditions have to be satisfied. Without a clear understanding of the conditions one may make false interpretations of the observation.

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The Aharonov–Bohm phase is always topological in its nature as long as the path of the charged particle encloses some magnetic flux. For the AC and HMW phases topological phases can develop only under certain circumstances. Essential conditions are that there is translational symmetry in one direction, and that the dipole moment in question is normal to both the direction of motion and the field with which it interacts. For the HMW phase it is particularly important to understand the constraints that must be satisfied by the electromagnetic fields through which the induced dipole is travelling. In the following we clarify these conditions which have not been accurately discussed for these induced dipoles. Previous discussions have been based on non-relativistic Lagrangian for interaction of a dipole with electromagnetic fields. In this work we give, for the first time, the relativistic Lagrangian density describing the motion of induced electric (and magnetic) dipoles in electric (magnetic) fields which induce the dipole, and magnetic (electric) fields which generate the HMW (AC) topological phases. We then use this relativistic Lagrangian density to derive the complete set of conditions for producing topological phases with induced dipoles in the non-relativistic regime. These conditions have been incompletely discussed in previous studies. However they have been met by the experiments which observed the HMW phase.

Because of its practical importance, we will first discuss the HMW phase conditions, and then discuss the case of AC phase.

The analysis should be based on induced dipoles from the start, which is what Wei, Han and Wei actually did [13]. They based their analysis, as did Wilkens [4], on the effective electric field seen by a particle moving in an magnetic field — called the Röntgen field $\mathbf{E}_R = \mathbf{v}\times\mathbf{B}$, which is added to the laboratory electric field $\boldsymbol{E}$ to obtain the electric field $\boldsymbol{E}_0 = (\mathbf{E}+\mathbf{v}\times\mathbf{B})$ experienced by the dipole in its rest frame. Through out this paper, $c=1$ and $\hbar=1$ units will be used. If the polarisability of the atom is α its electric dipole moment is

$$\mathbf{d} = \alpha(\mathbf{E}+\mathbf{v}\times\mathbf{B})\,. \tag{1}$$

The Lagrangian is then

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}m\mathbf{v}^2 + \frac{1}{2}\alpha(\mathbf{E}+\mathbf{v}\times\mathbf{B})^2 \\ &= \frac{1}{2}(m+\alpha\mathbf{B}^2)\mathbf{v}^2 + \frac{1}{2}\alpha\left(2\mathbf{v}\cdot(\mathbf{B}\times\mathbf{E}) + \mathbf{E}^2 - \alpha(\mathbf{v}\cdot\mathbf{B})^2\right)\,. \end{aligned} \tag{2}$$

Wei, Han, and Wei obtain their form of the HMW phase by neglecting the terms $\alpha\mathbf{E}^2$, verifying that $\alpha(\mathbf{B})^2 \ll m$, so that it may also be neglected, and ensuring that the experimental configuration is such that $\mathbf{v}\perp\mathbf{B}$, so that $\mathbf{v}\cdot\mathbf{B}$. We follow their example and the resulting Schrödinger equation is

$$\frac{1}{2m}\left(-i\nabla - \alpha(\mathbf{B}\times\mathbf{E})\right)^2\psi = 0\,, \tag{3}$$

which can be transformed to the field free Schrödinger equation by a phase transformation with the phase

$$\chi_{WHW} = \alpha\int_{\mathcal{C}} \mathbf{B}\times\mathbf{E}\cdot\mathbf{dr}\,. \tag{4}$$

This is a topological phase if

$$\text{curl}\,(\mathbf{B}\times\mathbf{E}) = \mathbf{B}\,\text{div}\,\mathbf{E} - \mathbf{E}\,\text{div}\,\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{E} - (\mathbf{E}\cdot\nabla)\mathbf{B}\,,$$

vanishes in the interference region and is non-zero in the excluded region. Now electric charges, as sources of $\mathbf{E}$, can generate the topological phase through the first term above [4]. Inducing the electric dipole removes the link to magnetic monopoles. So far we have not specified any condition on the electric field. Our previous relativistic analysis [3] suggests that the electric field should also be normal to the velocity. Why has this not come out in the analysis of Wei, Han, and Wei [13]?

In an attempt to answer this question, it would seem to be more reliable to obtain the relativistic corrections by taking the low velocity limit of a fully relativistic theory, rather than trying to introduce the corrections into the non-relativistic result. That is the approach we now adopt.

Minkowski [17] is the standard reference for the relativistic treatment of materials. There are accessible accounts in Pauli [18] and Møller [19]. Minkowski's proposal is that the relativistic version of $\mathbf{D}$ and $\mathbf{H}$ is the tensor $G_{\mu\nu}$ obtained by replacing $\mathbf{E}$ and $\mathbf{B}$ in $F_{\mu\nu}$ by $\mathbf{D}$ and $\mathbf{H}$. The Lagrangian density is then $-G_{\mu\nu}F^{\mu\nu}$ (up to some constant factor).

As (in Heaviside–Lorentz units)

$$\mathbf{D} = \mathbf{E}+\mathbf{P}\,,\;\; \mathbf{H} = \mathbf{B}-\mathbf{M}\,,$$

we should introduce a tensor (which Becker and Sauter [20] call the moments tensor) $K_{\mu\nu}$ constructed from $F_{\mu\nu}$ by replacing $\mathbf{E}$ with $\mathbf{P}$ and $\mathbf{B}$ with $-\mathbf{M}$. A moments tensor can be constructed from the electric and magnetic polarisation density of material bodies or the electric and magnetic dipole moments of individual atoms. We will use $K_{\mu\nu}$ as the moments tensor of atoms. This clearly shows that in relativity electric and magnetic moments get mixed up. This is nicely explained (with examples) by Becker and Sauter.

For now we will ignore intrinsic moments which are proportional to the spin of the particle, and consider only induced moments, which are proportional to the applied fields. We have to get to the generalisation of $\mathbf{P} = \alpha\mathbf{E}$, and $\mathbf{M} = \chi\mathbf{B}$, where α is the electric polarisability and χ is the magnetic susceptibility, which hold in the rest frame of the material. Following Minkowski we write, with u_μ as the four velocity of the moving particle,

$$u^\mu K_{\mu\nu} = \alpha u^\mu F_{\mu\nu} \qquad \text{and} \qquad u^\mu \tilde{K}_{\mu\nu} = \chi u^\mu \tilde{F}_{\mu\nu}, \tag{5}$$

which are identical to the above in the rest frame, and are tensor equations, so they are the correct generalisation.

In the rest frame the electric and magnetic fields are the spatial components of

$$E^\mu = u_\nu F^{\mu\nu} \quad \text{and} \quad B^\mu = u_\nu \tilde{F}^{\mu\nu}\,, \tag{6}$$

and, in the sense that qE^μ and gB^μ are the Minkowski four-force on a test charge q and a test magnetic monopole g in a frame in which they are moving with four-velocity u^μ, E^μ and B^μ are appropriate relativistic generalisations of $\mathbf{E}$ and $\mathbf{B}$ respectively. In particular qE^μ is simply the Minkowski four-force version of the Lorentz force on the charged particle. Its spatial component $q\gamma(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})$ is the Lorentz force on the charge, and its time component $q\gamma\,\boldsymbol{v}\cdot\boldsymbol{E}$ is the work done on the charged particle. Here $\gamma = 1/\sqrt{1-v^2}$.

Similarly

$$P^\mu = u_\nu K^{\mu\nu} \quad \text{and} \quad M^\mu = u_\nu \tilde{K}^{\mu\nu}\,, \tag{7}$$

are appropriate relativistic generalisations of $\mathbf{P}$ and $\mathbf{M}$ respectively.

We re-write equation (5) as

$$P^\mu = \alpha E^\mu \qquad \text{and} \qquad M^\mu = \chi B^\mu\,, \tag{8}$$

as the relationship between the induced electric moment and the electric field and the induced magnetic moment and the magnetic field. In the rest frame they are identical to the non-relativistic equations, and are tensor equations, so they are the correct generalisation, at least in the simplified situation that the polarisability and the susceptibility are scalars. Pauli's discussion (in §34) suggests that this is an adequate approximation, and we will use it.

Møller shows how to construct $K_{\mu\nu}$ from P^μ and M^μ, and hence the electromagnetic field. The result is