Physics Letters A ••• (••••) •••–•••

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

On weakly singular and fully nonlinear travelling shallow capillary–gravity waves in the critical regime

Dimitrios Mitsotakis [a], Denys Dutykh [b,*], Aydar Assylbekuly [c], Dauren Zhakebayev [d]

[a] Victoria University of Wellington, School of Mathematics, Statistics and Operations Research, PO Box 600, Wellington 6140, New Zealand
[b] LAMA, UMR 5127 CNRS, Université Savoie Mont Blanc, Campus Scientifique, F-73376 Le Bourget-du-Lac Cedex, France
[c] Khoja Akhmet Yassawi International Kazakh–Turkish University, Faculty of Natural Science, Department of Mathematics, 161200 Turkestan, Kazakhstan
[d] Al-Farabi Kazakh National University, Faculty of Mechanics and Mathematics, Department of Mathematical and Computer Modelling, 050000 Almaty, Kazakhstan

ARTICLE INFO

Article history:
Received 7 November 2016
Accepted 23 March 2017
Available online xxxx
Communicated by A.P. Fordy

Keywords:
Capillary–gravity waves
Peakons
Nonlinear dispersive waves

ABSTRACT

In this Letter we consider long capillary–gravity waves described by a fully nonlinear weakly dispersive model. First, using the phase space analysis methods we describe all possible types of localized travelling waves. Then, we especially focus on the critical regime, where the surface tension is exactly balanced by the gravity force. We show that our long wave model with a critical Bond number admits stable travelling wave solutions with a singular crest. These solutions are usually referred to in the literature as *peakons* or *peaked solitary waves*. They satisfy the usual speed-amplitude relation, which coincides with Scott–Russel's empirical formula for solitary waves, while their decay rate is the same regardless their amplitude. Moreover, they can be of depression or elevation type independent of their speed. The dynamics of these solutions are studied as well.

1. Introduction

In this Letter we investigate further the problem of hydrodynamic wave propagation over a horizontal impermeable bottom while we focus on a very particular regime of long capillary–gravity waves. Consider a two-dimensional Cartesian coordinate system Oxy where its horizontal axis coincides with the still water level $y = 0$. A layer of a perfect incompressible fluid is bounded from below by a flat impermeable bottom $y = -d$ and from above by the free surface $y = \eta(x, t)$. The fluid density is assumed to be constant $\rho > 0$. The total water depth is denoted by $h(x, t) \stackrel{\text{def}}{:=} d + \eta(x, t)$. Since the bottom is flat (*i.e.* $d = \text{const}$), we can equivalently replace the derivatives of the free surface elevation by the same derivatives of the total water depth, *i.e.* $\eta_t \equiv h_t$, $\eta_x \equiv h_x$. We shall use this property below.

In this derivation we follow the main lines of our previous work [8]. According to the Young–Laplace law, the pressure p jump across the interface is given by the following relation:

$$[\![p]\!] = -\sigma \left[\frac{\eta_x}{\sqrt{1 + \eta_x^2}} \right]_x = -\sigma \, \frac{\eta_{xx}}{\left(1 + \eta_x^2\right)^{3/2}} ,$$

where σ represents the surface tension. In this study we apply the small (free surface's) slope approximation to obtain $[\![p]\!] \approx -\sigma\, \eta_{xx}$. This pressure jump appears in the water wave problem through the Cauchy–Lagrange integral which serves as the dynamic boundary condition. Below we shall return to the surface tension effects by considering their potential energy since it allows to achieve easier our goals.

In order to derive model equations for gravity–capillary surface water waves we have to choose an ansatz to flow's structure and compute the system energy. Usually these model equations can be obtained if the horizontal velocity $u(x, y, t)$ is approximated by the depth-averaged fluid velocity $\bar{u}(x, t)$, and the vertical velocity is chosen to satisfy identically the incompressibility and bottom impermeability:

$$u(x,\ y,\ t) \approx \bar{u}(x,\ t)\,, \qquad v(x,\ y,\ t) \approx -(y + d)\, \bar{u}_x(x,\ t)\,.$$

Below we shall omit over bars in the notation since we work only with the depth-averaged velocity.

The various forms of energies for the specific fluid flow are estimated bellow: The kinetic energy $\mathcal{K}$ consists of the hydrostatic and non-hydrostatic corrections:

* Corresponding author.
E-mail addresses: dmitsot@gmail.com (D. Mitsotakis), Denys.Dutykh@univ-savoie.fr (D. Dutykh), asylbekuly@mail.ru (A. Assylbekuly), daurjaz@mail.ru (D. Zhakebayev).
URLs: http://dmitsot.googlepages.com/ (D. Mitsotakis), http://www.denys-dutykh.com/ (D. Dutykh).

http://dx.doi.org/10.1016/j.physleta.2017.03.041
0375-9601/

$$\mathcal{K} = \int_{t_1}^{t_2}\int_{x_1}^{x_2} \rho \left[\frac{h u^2}{2} + \frac{h^3 u_x^2}{6} \right] \mathrm{d}x \, \mathrm{d}t \, .$$

The potential energy consists of the gravity

$$\mathcal{V}_g = \int_{t_1}^{t_2}\int_{x_1}^{x_2} \frac{\rho g h^2}{2} \, \mathrm{d}x \, \mathrm{d}t \, ,$$

and capillary contributions:

$$\mathcal{V}_c = \int_{t_1}^{t_2}\int_{x_1}^{x_2} \rho \tau \left[\sqrt{1 + h_x^2} - 1 \right] \mathrm{d}x \, \mathrm{d}t \approx \frac{1}{2} \int_{t_1}^{t_2}\int_{x_1}^{x_2} \rho \tau h_x^2 \, \mathrm{d}x \, \mathrm{d}t \, ,$$

to which we applied the small slope approximation (and we introduced another physical constant $\tau \stackrel{\mathrm{def}}{:=} \frac{\sigma}{\rho}$). For more details on the derivation of energies $\mathcal{K}$ and $\mathcal{V}_g$ we refer to [8]. Now we can assemble the action integral:

$$\mathscr{S} \stackrel{\mathrm{def}}{:=} \mathcal{K} - \mathcal{V}_g - \mathcal{V}_c + \int_{t_1}^{t_2}\int_{x_1}^{x_2} \rho \left[h_t + [h u]_x \right] \phi \, \mathrm{d}x \, \mathrm{d}t \, ,$$

where we enforced the mass conservation by introducing a Lagrange multiplier $\phi(x, t)$. By applying the Hamilton–Ostrogradsky variational principle and eliminating the Lagrange multiplier $\phi(x, t)$ from the equations, we arrive at the following system of equations:

$$h_t + [h u]_x = 0 \, , \tag{1.1}$$

$$u_t + u u_x + g h_x = \frac{1}{3h} \left[h^3 (u_{xt} + u u_{xx} - u_x^2) \right]_x + \tau h_{xxx} . \tag{1.2}$$

These are the celebrated Serre–Green–Naghdi (SGN) equations with weak[1] surface tension effects, [10,14]. The full list of (physical) conservation laws is given below. The mass conservation was already given in equation (1.1). The remaining identities are given below:

$$\left[u - \frac{(h^3 u_x)_x}{3h} \right]_t + \left[\frac{u^2}{2} + g h - \frac{h^2 u_x^2}{2} - \frac{u (h^3 u_x)_x}{3h} - \tau h_{xx} \right]_x = 0 \, , \tag{1.3}$$

$$[h u]_t + \left[h u^2 + \tfrac{1}{2} g h^2 + \tfrac{1}{3} h^2 \gamma - \tau \mathscr{R} \right]_x = 0 \, , \tag{1.4}$$

where we introduced for the sake of notation compactness two quantities:

$$\gamma \stackrel{\mathrm{def}}{:=} h \left[u_x^2 - u_{xt} - u u_{xx} \right] ,$$

$$\mathscr{R} \stackrel{\mathrm{def}}{:=} h h_{xx} - \tfrac{1}{2} h_x^2 \, .$$

The quantity γ has a physical sense of the vertical acceleration of fluid particles computed at the free surface. Some of the conservation laws shall be used below to study travelling waves to the SGN system (1.1)–(1.2).

The conservation of energy can be written in the form:

$$\mathscr{H}_t + \mathscr{Q}_x = 0 \, , \tag{1.5}$$

where

$$\mathscr{H} \stackrel{\mathrm{def}}{:=} \frac{hu^2}{2} + \frac{h^3 u_x^2}{6} + \frac{gh^2}{2} + \frac{\tau}{2} h_x^2 \, ,$$

$$\mathscr{Q} \stackrel{\mathrm{def}}{:=} \left(\frac{u^2}{2} + \frac{h^2 u_x^2}{6} + gh + \frac{h\gamma}{3} - \tau h_{xx} \right) hu + \tau h_x (hu)_x \, ,$$

with $\mathscr{H}$ the approximation of the total energy and $\mathscr{Q}$ the energy flux.

The nature of the solutions of the SGN system depend on the parameter τ and they have been studied for the values of $\tau \neq 1/3$. It is know that for $\tau < 1/3$ the system admits classical solitary waves of elevation while for $\tau > 1/3$ there are only solitary waves of depression. In the rest of the paper we study in detail the travelling wave solutions of the SGN equation with special emphasis in the critical case $\tau = 1/3$.

2. Travelling wave solutions

In this Section we focus on a special class of solutions — the so-called travelling waves. The main simplifying circumstance is that the flow becomes steady in the frame of reference moving with the wave. Thus, it allows to analyze Ordinary Differential Equations (ODEs) instead of working with Partial Differential Equations (PDEs). We substitute the following solution *ansatz* into all equations:

$$u(x, t) = u(\xi) \, , \quad h(x, t) = h(\xi) \, , \quad \xi \stackrel{\mathrm{def}}{:=} x - c t \, ,$$

where $c > 0$ is the wave speed.[2] Moreover, we focus on localized solutions of this type — the so-called *solitary waves*. They satisfy the following boundary conditions:

$$h^{(n)}(\xi) \to 0 \, , \qquad u^{(n)}(\xi) \to 0 \, , \qquad \text{as} \quad \xi \to \infty \, ,$$

$n = 1, 2, \ldots$. For the total depth and horizontal velocity profiles (*i.e.* $n = 0$) we have the following boundary conditions:

$$h(\xi) \to d \, , \quad u(\xi) \to -c \, , \qquad \text{as} \quad \xi \to \infty \, .$$

The mass conservation equation (1.1) readily yields a relation between u and h:

$$u(\xi) = -\frac{c d}{h(\xi)} \, . \tag{2.1}$$

By substituting these relations into the conservation laws (1.3), (1.4) and taking a linear combination of these two equations leads to the following *implicit* ODE $\mathcal{E}(h', h) = 0$ for the total water depth:

$$\mathcal{E}(h', h) \stackrel{\mathrm{def}}{:=} \frac{\mathrm{Fr}\,(h')^2}{3} - \mathrm{Bo}\,(h')^2 \frac{h}{d} - \mathrm{Fr} + \frac{(2\,\mathrm{Fr} + 1)\, h}{d} - \frac{(\mathrm{Fr} + 2)\, h^2}{d^2} + \frac{h^3}{d^3} = 0 \, , \tag{2.2}$$

where we introduced two dimensionless numbers:

- $\mathrm{Fr} \stackrel{\mathrm{def}}{:=} \frac{c^2}{g d}$: the Froude (also known as Eötvös) number
- $\mathrm{Bo} \stackrel{\mathrm{def}}{:=} \frac{\tau}{g d^2} \equiv \frac{\sigma}{\rho g d^2}$: the Bond number.

The details on the derivation of the master equation (2.2) with the full surface tension term can be found in [7]. Let us compute the partial derivatives of the function $\mathcal{E}(h', h)$:

[1] The word 'weak' comes from the fact that we applied small slope approximation.

[2] In other words, we consider waves moving in the rightward direction, without loosing generality.