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Analytical solutions for the radial Scarf II potential

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The real Scarf II potential is discussed as a radial problem. This potential has been studied extensively as a one-dimensional problem, and now these results are used to construct its bound and resonance solutions for $l = 0$ by setting the origin at some arbitrary value of the coordinate. The solutions with appropriate boundary conditions are composed as the linear combination of the two independent solutions of the Schrödinger equation. The asymptotic expression of these solutions is used to construct the $S_0(k)$ s-wave S-matrix, the poles of which supply the k values corresponding to the bound, resonance and anti-bound solutions. The location of the discrete energy eigenvalues is analyzed, and the relation of the solutions of the radial and one-dimensional Scarf II potentials is discussed. It is shown that the generalized Woods–Saxon potential can be generated from the Rosen–Morse II potential in the same way as the radial Scarf II potential is obtained from its one-dimensional correspondent. Based on this analogy, possible applications are also pointed out.

1. Introduction

Exactly solvable quantum mechanical potentials proved to be invaluable tools in the understanding of many fundamental quantum mechanical concepts. In particular, they give insight into complex phenomena, like the symmetries of quantum mechanical systems, and they allow the investigation of transitions through critical parameter domains. Besides this, analytical solutions serve as a firm basis for the development of numerical techniques.

The one-dimensional Schrödinger equation

$$-\psi''(x) + V(x)\psi(x) = E\psi(x) \tag{1}$$

occurs in many applications. Here the potential function and the energy eigenvalue are defined such that they contain reduced mass m and $\hbar$ as $V(x) = 2mv(x)/\hbar^2$ and $E = 2m\epsilon/\hbar^2$, so their physical dimension is distance^{-2}. In the simplest case (1) is defined on the full x axis, i.e. $x \in (-\infty, \infty)$, while for spherical potentials defined in higher, typically three dimension, Eq. (1) can be obtained after the separation of the angular variables, if only the s-wave ($l = 0$) solutions are considered. In this case the problem is defined on the positive half axis, $r \in [0, \infty)$, and the x variable is denoted by r. Besides these options, (1) can also be defined on finite sections of the real x axis, or even on more complicated trajectories of the complex x plane, but we shall not consider these in the present work.

Being a second-order ordinary differential equation, (1) has two independent solutions, and the physical solutions can be obtained as linear combination of these, satisfying the appropriate boundary conditions. Due to normalizability, bound states have to vanish at the boundaries (i.e. $x = \pm\infty$ in one dimension, and $r = 0$ and $r = \infty$ in the radial case). Unbound solutions, e.g. scattering and resonance solutions also have to satisfy asymptotic boundary conditions, depending on the nature of the potential. If $V(x)$ vanishes exactly or exponentially for $x \to \pm\infty$, then these solutions of the one-dimensional problem have exponential asymptotic components $\exp(\pm ikx)$, where $E = k^2$. In the radial case the same asymptotics are valid for $r \to \infty$, while for $r = 0$ these solutions have to vanish.

There are some potentials that are defined both as one-dimensional and as radial problems, e.g. the harmonic oscillator. The bound-state solutions of these two problems are related to each other in a special way: the odd wave functions of the one-dimensional potential, which vanish at $x = 0$, are identical for $x \geq 0$ to the s-wave ($l = 0$) radial wave functions, and the energy eigenvalues are also identical.

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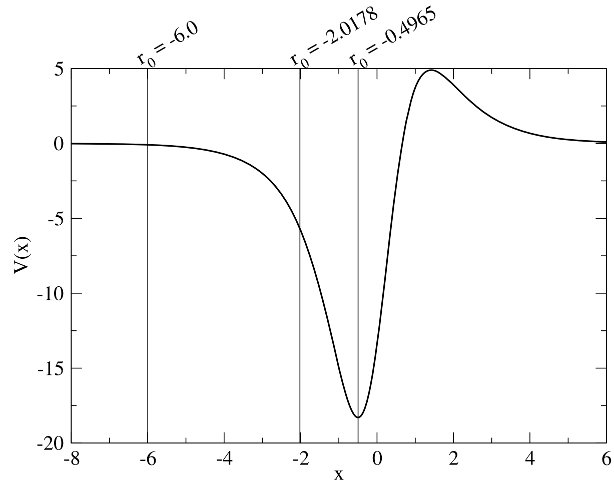

Fig. 1. The structure of the one-dimensional Scarf II potential for $V_1 = 13.4$ and $V_2 = 18.92$ corresponding to $c = 1$, $\alpha = -4.3 - 2.2\mathrm{i} = \beta^*$, $s = 3.8$ and $\lambda = 2.2$. This potential has four bound states at $E_0 = -14.44$, $E_1 = -7.84$, $E_2 = -3.24$ and $E_3 = -0.64$. The vertical lines at $x = r_0 = -0.4965$, -2.017825 (abbreviated in the plot) and -6.0 define three radial potentials (see Sec. 3) with different location of the origin. In the first case $r_0 = x_-$, while the second r_0 corresponds to the first node of $\psi_2(x)$ in (5).

A rather effective way for the unified discussion of bound, scattering and resonance solutions in asymptotically vanishing potentials is the application of the transmission coefficient $T(k)$ (in one dimension) and the s-wave S matrix $S_0(k)$ (in the radial case). These quantities can be constructed from the asymptotic solutions, and their poles correspond to the bound, anti-bound and resonance states. From the exact solutions of these problems $T(k)$ and $S_0(k)$ can also be expressed in closed analytic form.

Here we discuss the Scarf II potential as a radial problem. This potential has two independent terms and belongs to the shape-invariant [1] subclass of the Natanzon potential class [2], which contains problems with bound-state solutions written in terms of a single hypergeometric function. The first reference to potential (2) in the English literature occurred in 1983 in Ref. [1], so it is sometimes referred to as the Gendenshtein potential. However, it was already mentioned a year before in a Russian monograph [3]. Its detailed description was presented later, e.g. the normalization coefficients of its bound-state solutions have been calculated only recently [4]. The transmission and reflection coefficients have been given in Ref. [5], with corrections added in Ref. [6]. It has been a favorite toy model in $\mathcal{PT}$-symmetric quantum mechanics, where it was used to demonstrate the breakdown of $\mathcal{PT}$ symmetry [7,8]. Further studies concerned its algebraic [9,6] and scattering aspects [6], the combined effects of SUSYQM and $\mathcal{PT}$ symmetry [10], the pseudo-norm of its bound states [4], the handedness (chirality) effects in scattering [11], spectral singularities [12], unidirectional invisibility [13] and the accidental crossing of its energy levels [14].

Despite its prominent status as a one-dimensional quantum system, the Scarf II potential has not been considered yet as a radial problem. Here we fill this gap by introducing a lower cut at a certain $x = r_0$ value and prescribing the appropriate boundary conditions. We construct the S-matrix for the s-wave solutions, $S_0(k)$, and determine its poles on the complex k plane to identify its bound, anti-bound and resonance solutions. This will be done in Sec. 3, following the discussion of the one-dimensional problem for reference in Sec. 2. In Sec. 4 the analogy with the case of the generalized Woods–Saxon and the Rosen–Morse II potentials will be outlined, and possible applications are pointed out. Finally the results are summarized in Sec. 5.

2. The Scarf II potential in one dimension

A possible parametrization of this potential is [10]

$$V(x) = -\frac{V_1}{\cosh^2(cx)} + \frac{V_2 \sinh(cx)}{\cosh^2(cx)}, \tag{2}$$

where

$$V_1 = c^2\left(\frac{\alpha^2+\beta^2}{2} - \frac{1}{4}\right), \qquad V_2 = \mathrm{i}c^2\frac{\beta^2-\alpha^2}{2}, \tag{3}$$

and $c > 0$ is a scaling factor of the coordinate. This potential is real if $\alpha^* = \beta$ holds, while it is $\mathcal{PT}$-symmetric if α and β are real or imaginary. In what follows we consider the real version only. Potential (2) is depicted in Fig. 1 for some values of the parameters. It has a minimum x_- and a maximum x_+ at

$$x_\pm = c^{-1}\sinh^{-1}\left[\frac{V_1}{V_2} \pm \left[\left(\frac{V_1}{V_2}\right)^2 + 1\right]^{1/2}\right]. \tag{4}$$

The potential reflected by $x = 0$ can be constructed easily by considering $V_2 \to -V_2$, i.e. $\alpha \leftrightarrow \beta$.

The bound-state wave functions are

$$\psi_n(x) = C_n(1 - \mathrm{i}\sinh(cx))^{\frac{\alpha}{2}+\frac{1}{4}}(1 + \mathrm{i}\sinh(cx))^{\frac{\beta}{2}+\frac{1}{4}} P_n^{(\alpha,\beta)}(\mathrm{i}\sinh(cx)), \tag{5}$$

while the corresponding energy eigenvalues are written as