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Minimal sets of dequantizers and quantizers for finite-dimensional quantum systems

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ABSTRACT

The problem of finding and characterizing minimal sets of dequantizers and quantizers applied in the mapping of operators onto functions is considered, for finite-dimensional quantum systems. The general properties of such sets are determined. An explicit description of all the minimum self-dual sets of dequantizers and quantizers for a qubit system is derived.

1. Introduction

The phase-space formulation of quantum mechanics is still in the focus of research interest, as it has numerous important applications [1–4]. Quasi-probability distributions such as the Wigner function [5], the Husimi Q-function [6,7] and Glauber–Sudarshan P-function [8,9] describe completely the states of a quantum system and they are widely used for calculations in various physical problems [10–15]. They have proven to be very useful in quantum optics [16–18]. A probability representation with fair probability distributions defined on the phase space has also been introduced in the literature [19–21]. A probability distribution called the symplectic tomogram was introduced in connection with measuring the quantum states of light by means of optical homodyne tomography [22–24]. The properties of this tomographic probability representation are discussed in detail in review [25].

In order to use quasi-probability distributions and tomograms in physical problems the operators modeling observable physical quantities have to be represented. [26]. This representation is called the symbol of operators. The algebra of symbols corresponding all possible manipulations with operators on the Hilbert space can be constructed by applying the general star-product scheme [27–29]. Within this formalism one can relate operators to their symbols using dequantizers and can reconstruct operators from their symbols using quantizers. The relations between different phase-space representations can be also determined in this framework [29–32].

All these ideas can be extended to finite dimensional quantum systems. Finding a complete, continuous Wigner function for such system is still a subject of investigations [33–35]. Beside these efforts there is an increasing interest in the construction of discrete phase spaces and Wigner functions owing to their possible applications in quantum information science. There are several ways of constructing such a phase space and the definition of a discrete Wigner function in this space is still ambiguous [36–47]. The approach introduced in [40] has proven to be well suited to study various quantum information problems [48]. In this method, an $N \times N$ phase space is defined for N dimensional quantum systems, where N is a power of a prime number. This is the case, e.g. for qubit systems. This phase space has the same geometric properties as those of the ordinary infinite dimensional phase space. Wigner functions can be defined in this space using Hermitian operators connected to special mutually orthogonal sets of parallel lines called striations. There exist $N+1$ different striations and the bases associated with them are mutually unbiased [4,43,49,50]. Such discrete Wigner functions have the same essential properties as their continuous counterparts. The most interesting one from the point of view of tomographic measurements is that the sum of

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values of a Wigner function along any line in phase space is equal to the probability of detecting the basis state associated with the line [48].

Tomographic probability distributions called spin tomograms [51–54], and unitary matrix tomograms [55] have been also developed for finite dimensional spin systems. The star product formalism of symbols for N-dimensional systems is described in detail in [56]. Using this formalism the relations between tomograms and Wigner functions for one and two qubits have been determined [57,58,56].

In this paper we consider the problem of finding and characterizing minimal sets of quantizers and dequantizers for finite dimensional quantum systems. We determine the general properties of such sets. Given minimal sets of dequantizers and quantizers for a particular quantum system, any type of symbols of the operators and the quantum states consisting of minimal elements, e.g., discrete Wigner functions, can be treated in a common framework. We find explicit expressions describing all the minimal self-dual sets of dequantizers and quantizers for a qubit system.

The paper is organized as follows. In Section 2 we present the general formalism of mapping operators onto functions based on dequantizers and quantizers. The general properties of minimal sets of dequantizers and quantizers for N dimensional systems is described in Section 3. In Section 4 the explicit form of all minimal self-dual sets of dequantizers and quantizers for a qubit system is found.

2. Dequantizers and quantizers

In this section we summarize the general formalism of using c-number functions instead of operators to describe quantum systems [26–29]. Let $\hat{A}$ be a Hermitian operator acting on a Hilbert space $\mathcal{H}$ so it can be an operator describing an observable or the density operator $\hat{\rho}$ of the quantum system. Suppose we have a set of linear operators $\hat{U}(x)$ acting on $\mathcal{H}$ and labelled by the parameter x that is an n-dimensional vector $x = (x_1, x_2, \ldots, x_n)$ in the general case. One can construct a c-number function $f_{\hat{A}}(x)$ called the symbol of the operator $\hat{A}$ using the definition

$$f_{\hat{A}}(x) = \mathrm{Tr}[\hat{A}\hat{U}(x)]. \quad (1)$$

This linear mapping of operators onto functions is invertible if there is a set of operators $\hat{D}(x)$ acting on $\mathcal{H}$ such that

$$\hat{A} = \int f_{\hat{A}}(x)\hat{D}(x)\mathrm{d}x. \quad (2)$$

The operators $\hat{U}(x)$ and $\hat{D}(x)$ are called dequantizers and quantizers, respectively. In this formalism the operation for functions corresponding to the multiplication of $\hat{A}$ and $\hat{B}$ is called star product and defined by

$$f_{\hat{A}\hat{B}}(x) = f_{\hat{A}}(x) * f_{\hat{B}}(x) = \mathrm{Tr}[\hat{A}\hat{B}\hat{U}(x)]. \quad (3)$$

Multiplying Eq. (2) by the operator $\hat{U}(x')$ and taking the trace we get

$$f_{\hat{A}}(x') = \int f_{\hat{A}}(x)\,\mathrm{Tr}[\hat{D}(x)\hat{U}(x')]\mathrm{d}x. \quad (4)$$

For continuous systems the operators $\hat{U}(x)$ are defined in the usual phase space with the coordinates (q, p) while for discrete systems x can be both discrete and continuous as in the case of spin tomograms, or it can be purely discrete as in the case of discrete Wigner functions defined e.g. in [40]. In the latter case Eqs. (2) and (4) can be written as

$$\hat{A} = \sum_{k=1}^{N} f_{\hat{A}}(k)\hat{D}(k) \quad (5)$$

and

$$f_{\hat{A}}(k') = \sum_{k=1}^{N} f_{\hat{A}}(k)\,\mathrm{Tr}[\hat{D}(k)\hat{U}(k')], \quad (6)$$

respectively.

For a d dimensional discrete quantum system the term minimal set of quantizers and dequantizers is introduced for sets containing d^2 linearly independent operators. From Eq. (6) it follows that the quantizer and dequantizer operators of such sets satisfy the condition

$$\mathrm{Tr}\left(\hat{D}(k)\hat{U}(k')\right) = \delta(k, k'). \quad (7)$$

For some special set of dequantizers the symbols are called the Wigner function [40]. These dequantizers are Hermitian operators and coincide with the corresponding quantizers. So they form a self-dual system.

3. Minimal sets of dequantizers and quantizers

In this section we consider the general properties of minimal sets of quantizers and dequantizers for N-dimensional systems.

Let us analyze first a two-dimensional qubit system. For this system the minimal set of dequantizers consists of four linearly independent operators $\hat{U}^{(k)}$ that can be represented by four matrices

$$\hat{U}^{(k)} = \begin{pmatrix} U_{11}^{(k)} & U_{12}^{(k)} \\ U_{21}^{(k)} & U_{22}^{(k)} \end{pmatrix}, \qquad k = 1, 2, 3, 4. \quad (8)$$

First we address the problem of determining the four corresponding quantizers

$$\hat{D}^{(k)} = \begin{pmatrix} D_{11}^{(k)} & D_{12}^{(k)} \\ D_{21}^{(k)} & D_{22}^{(k)} \end{pmatrix}, \qquad k = 1, 2, 3, 4 \quad (9)$$

satisfying Eq. (7) that can be written using the notations of Eqs. (8) and (9) as

$$\mathrm{Tr}\left(\hat{U}^{(k)}\hat{D}^{(k')}\right) = \delta(k, k'). \quad (10)$$

We assume that the dequantizers $U^{(k)}$ are known.

Let us introduce the operator

$$\hat{A} = \begin{pmatrix} U_{11}^{(1)} & U_{21}^{(1)} & U_{12}^{(1)} & U_{22}^{(1)} \\ U_{11}^{(2)} & U_{21}^{(2)} & U_{12}^{(2)} & U_{22}^{(2)} \\ U_{11}^{(3)} & U_{21}^{(3)} & U_{12}^{(3)} & U_{22}^{(3)} \\ U_{11}^{(4)} & U_{21}^{(4)} & U_{12}^{(4)} & U_{22}^{(4)} \end{pmatrix}, \qquad m, n = 1, 2, 3, 4 \quad (11)$$

built up from the elements of the four dequantizer operators and the operator

$$\hat{B} = \begin{pmatrix} D_{11}^{(1)} & D_{12}^{(1)} & D_{21}^{(1)} & D_{22}^{(1)} \\ D_{11}^{(2)} & D_{12}^{(2)} & D_{21}^{(2)} & D_{22}^{(2)} \\ D_{11}^{(3)} & D_{12}^{(3)} & D_{21}^{(3)} & D_{22}^{(3)} \\ D_{11}^{(4)} & D_{12}^{(4)} & D_{21}^{(4)} & D_{22}^{(4)} \end{pmatrix} \quad (12)$$

containing the elements of the four quantizer operators. It is easy to see that the equation (10) is equivalent to

$$\hat{A}\hat{B}^{T} = \hat{I} \quad (13)$$

As the operators $\hat{U}^{(k)}$ are linearly independent therefore the determinant of the matrix $\hat{A}$ is not equal to zero. From Eq. (13) it