



Symmetries and singularities of the Szekeres system



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ABSTRACT

The Szekeres system is studied with two methods for the determination of conservation laws. Specifically we apply the theory of group invariant transformations and the method of singularity analysis. We show that the Szekeres system admits a Lagrangian and the conservation laws that we find can be derived by the application of Noether's theorem. The stability for the special solutions of the Szekeres system is studied and it is related with the Left or Right Painlevé Series which describes the expansions.

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Gravitational models with vanishing magnetic part of Weyl curvature tensor and irrotational dust fluid component have been studied by Bruni, Matarrese and Pantano in [1]. Because there is no information dissemination with gravitational or sound waves between the worldlines of neighbouring fluid elements these models are called silent universe [2]. The field equations form a system of six first-order ordinary differential equations. The exact solutions at the critical points were found to be Tolman–Bondi, Kantowski–Sachs or Szekeres geometries which can be seen as perturbations of Friedmann–Lemaître–Robertson–Walker spacetimes [1]. The stability of these solutions was studied in [3] where it was found that solutions which describe pancakelike collapse are stable while those that describe spindlelike collapse are unstable, while in [4] was shown that the spacetimes that follow from the Szekeres system are “partially” locally rotational spacetimes (PLRS). The integrability conditions for the silent universes were studied in [2] wherein a conjecture was given such that “there are no spatially inhomogeneous irrotational dust silent models, whose Weyl curvature tensor is of algebraic Petrov type I”.

The field equations of the silent universe comprise a system of six first-order ordinary differential equations¹ in which the de-

pendent variables are: the energy density of the dust fluid, ρ , the expansion rate of the observer, θ , the shear components of the observer, σ_1 and σ_2 , and the two components of the electric part of the Weyl curvature tensor, i.e. E_1 and E_2 . A special case is when the electric parts of the Weyl tensor are equal, i.e., $E_1 = E_2$ as also the shear $\sigma_1 = \sigma_2$. In that case any irrotational dust model with vanishing magnetic field is described by the so-called Szekeres system [5,6]. The dynamics of the Szekeres system can be found in [7], while a covariant formulation based on the $1 + 1 + 2$ decomposition was derived recently in [8].

The equations which form the Szekeres system are

$$\dot{\rho} + \theta\rho = 0, \quad (1)$$

$$\dot{\theta} + \frac{\theta^2}{3} + 6\sigma^2 + \frac{1}{2}\rho = 0, \quad (2)$$

$$\dot{\sigma} - \sigma^2 + \frac{2}{3}\theta\sigma + \mathcal{E} = 0, \quad (3)$$

$$\dot{\mathcal{E}} + 3\mathcal{E}\sigma + \theta\mathcal{E} + \frac{1}{2}\rho\sigma = 0, \quad (4)$$

plus the algebraic equation

$$\frac{\theta^2}{3} - 3\sigma^2 + \frac{{}^{(3)}R}{2} = \rho, \quad (5)$$

where a dot means contraction of the covariant derivative ∇_μ , with respect to the timelike four-vector field u^μ , such that $\dot{A} = A_{;\mu}u^\mu$.

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¹ The system is an algebraic–differential system in which the algebraic equation is related with the curvature of the three-dimensional space.

However except from the above system the 1 + 3 analysis of the field equations provides the spacelike constraints [9]

$$h_{\mu}^{\nu} \sigma_{\nu;\alpha}^{\alpha} = \frac{2}{3} h_{\mu}^{\nu} \theta_{;\nu}, \quad h_{\mu}^{\nu} E_{\nu;\alpha}^{\alpha} = \frac{1}{3} h_{\mu}^{\nu} \rho_{;\nu} \quad (6)$$

where $h_{\mu\nu}$ is the decomposable tensor defined by the expression $h_{\mu\nu} = g_{\mu\nu} - \frac{1}{u_{\lambda} u^{\lambda}} u_{\mu} u_{\nu}$, and $E_{\nu}^{\mu} = E e_{\nu}^{\mu}$, $\sigma_{\nu}^{\mu} = \sigma e_{\nu}^{\mu}$. Obviously when the equations (6) are satisfied identically then equations (1)–(4) describe a system of ordinary differential equations, while in general they form a system of partial differential equations. The spacelike constraint equations are essential for the integrability of the silent models and in order the solution to reduce to the Szekeres models [2,9]. In the following we work with the system (1)–(4) and the constants of integration will be functions independent on the time derivative which are constrained by the system (6).

The integrability of the Szekeres system (1)–(4) has been proved recently with the use of the Darboux polynomial and the Jacobi multiplier methods [10]. Here we study the Szekeres system with two different methods in the search for integrability and the existence of analytical solutions. Specifically we use the symmetry method, group invariant transformations and the singularity analysis. The purpose of this analysis is to study the relationship between the two methods and see how the various solutions, different universes, are related with the symmetries or the movable singularities of the field equations.² Both methods have played an important role in gravitational studies while recently they have been applied in modified theories of gravity for the determinant of integrable field equations, for instance see [12–20] and references therein.

The jet-space which is defined by the Szekeres system is $J_S = \{\tau, \rho, \theta, \sigma, \mathcal{E}\}$ where τ is the independent variable and it is a local coordinate of the spacetime. Consider the generator X of a one-parameter point transformation in space J_S , then

$$X = \xi \left(\tau, Y^B \right) \partial_{\tau} + \eta^{\rho} \left(\tau, Y^B \right) \partial_{\rho} + \eta^{\theta} \left(\tau, Y^B \right) \partial_{\theta} + \eta^{\sigma} \left(\tau, Y^B \right) \partial_{\sigma} + \eta^{\mathcal{E}} \left(\tau, Y^B \right) \partial_{\mathcal{E}}, \quad (7)$$

where $Y^B = (\rho, \theta, \sigma, \mathcal{E})$.

However, the Szekeres system can be written as a system of two second-order ordinary differential equations. Without loss of generality we select the two dependent variables to be the energy density, ρ , and the electric component, $\mathcal{E}$.

The system of second-order differential equations is

$$\frac{d^2 \rho}{d\tau^2} = \Phi(\rho, \mathcal{E}) \quad \text{and} \quad (8)$$

$$\frac{d^2 \mathcal{E}}{d\tau^2} = \Psi(\rho, \mathcal{E}), \quad (9)$$

where now the expansion rate and the shear are given from the relations

$$\theta = -\frac{\dot{\rho}}{\rho} \quad \text{and} \quad (10)$$

$$\sigma = \frac{2(\dot{\rho}\mathcal{E} - \rho\dot{\mathcal{E}})}{\rho(\rho + 6\mathcal{E})}. \quad (11)$$

With the use of the latter expressions we observe that any point symmetry, (7), of the Szekeres system is nothing else than a generalised symmetry for the system, (9).

We now consider that the symmetry vector is the generator of a point transformation in $\bar{J}_S = \{\tau, \rho, \mathcal{E}\}$. Then

$$\bar{X} = \xi \left(\tau, Y^B \right) \partial_{\tau} + \eta^{\rho} \left(\tau, Y^B \right) \partial_{\rho} + \eta^{\mathcal{E}} \left(\tau, Y^B \right) \partial_{\mathcal{E}}. \quad (12)$$

Note that $\bar{J}_S \subset J_S$, which means that by assuming a point transformation at $\bar{J}_S$ we find only a partial number of symmetries for the original system, but, as well see below, that it is sufficient in order to prove the integrability of the system (9) as also to determine an analytical solution.

We find that the admitted Lie point symmetries of the second-order system (9) are

$$X_1 = \partial_{\tau},$$

$$X_2 = \frac{\rho \mathcal{E}}{\rho + 6\mathcal{E}} (6\partial_{\rho} - \partial_{\mathcal{E}})$$

and

$$X_3 = \tau \partial_{\tau} - \frac{2\rho(\rho + 12\mathcal{E})\partial_{\rho} + 12\mathcal{E}^2\partial_{\mathcal{E}}}{\rho + 6\mathcal{E}}$$

which constitute the $A_1 \otimes_S 2A_1$ Lie algebra. X_1 is the autonomous symmetry and X_3 is a rescaling symmetry which we see below that the existence of both is important for the singularity analysis.

We apply the coordinate transformation³ to (9),

$$\rho = \frac{6}{(1-x)y^3}, \quad \mathcal{E} = \frac{x}{y^3(x-1)}, \quad (13)$$

and the Szekeres system takes the simpler form,

$$\ddot{x} + 2\frac{\dot{y}}{y}\dot{x} - \frac{3}{y^3}x = 0, \quad (14)$$

and

$$\ddot{y} + \frac{1}{y^2} = 0. \quad (15)$$

From (15) we find the conservation law

$$I_1 = \dot{y}^2 - 2y^{-1}, \quad (16)$$

which gives

$$\int \frac{dy}{\sqrt{I_1 + 2y^{-1}}} = \tau - \tau_0$$

while, when $I_1 = 0$, we have the closed-form solution

$$y(\tau) = \left(\frac{3\sqrt{2}}{2} (\tau - \tau_0) \right)^{\frac{2}{3}}.$$

Therefore, when we substitute $y(\tau)$ into (14), we obtain a second-order linear equation which is maximally symmetric and integrable. In the limit in which $I_1 = 0$, we find the closed-form solution

$$x(t) = x_1 (\tau - \tau_0)^{\frac{2}{3}} + x_2 (\tau - \tau_0)^{-1}. \quad (17)$$

Moreover we observe that the Szekeres system, (14), (15), follows from the variation of the Action integral

$$S = \int L(x, \dot{x}, y, \dot{y}) d\tau \quad (18)$$

² A discussion of these two different methods for the study of integrability can be found in [11].

³ The new variables that we have considered are related with the q-scalar variables which have been introduced in [7]. Specifically it holds $y^3 = 6\rho_q^{-1}$, $x = \Delta^{(\rho)} (1 + \Delta^{(\rho)})^{-1}$, where by definition it follows that $\theta_q = 3\dot{y}y^{-1}$ and $\dot{x}(1-x)^{-1} = -3\sigma$.

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