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Completing the proof of "Generic quantum nonlocality" ☆

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ABSTRACT

In a paper by Popescu and Rohrlich [1] a proof has been presented showing that any pure entangled multiparticle quantum state violates some Bell inequality. We point out a gap in this proof, but we also give a construction to close this gap. It turns out that with some extra effort all the results from the aforementioned publication can be proven. Our construction shows how two-particle entanglement can be generated via performing local projections on a multiparticle state.

1. Introduction

The question which quantum states violate a Bell inequality and which not is of central importance for quantum information processing. In Ref. [1] it has been shown that any pure multiparticle quantum state violates a Bell inequality. The strategy for proving this statement was the following: First, one can show that for any entangled pure state on N particles one can find projective measurements on $N-2$ particles, such that for appropriate results of the measurements the remaining two particles are in an entangled pure state. Then, one can apply the known fact that any pure bipartite entangled state violates some Bell inequality [2].

In this note we point out a gap in the proof presented in Ref. [1]. The gap concerns the part where the projective measurements on $N-2$ particles are made. It turns out that a certain logical step does not follow from the previous statements and we give an explicit counterexample for a conclusion drawn at the critical point. Luckily it turns out, however, that with a significantly refined and extended argumentation the main statement can still be proven. Independently of the connection to Ref. [1] our results provide a constructive way how a two-particle entangled state can be generated from an N-particle state by performing local projections onto $N-2$ particles. This may be of interest for the theory of multiparticle entanglement.

This note is organized as follows. In Section 2 we discuss the proof from Ref. [1] and the problem with a Lemma used there. In Section 3 we present a detailed proof of the required statement for qubits. Finally, in Section 4 we discuss the higher-dimensional case as well as some other observations needed for the proof.

2. Discussion of the original argument

The gap concerns the proof of the Lemma on page 296 of Ref. [1]. This lemma states that:

Let $|\psi\rangle$ be an N system entangled state. For any two of the N systems, there exists a projection onto a direct product of state of the other $N-2$ systems, that leaves the two systems in an entangled state. In the following we show that while the Lemma is correct, there is a gap in its original proof. Doing so, in this section we will reformulate the proof in modern language in order to see where the problem is. For simplicity, we first consider only qubits.

The proof from Ref. [1] is a proof by contradiction, so it starts with assuming the opposite. So, orthogonal basis vectors $|b_i\rangle \in \{|0\rangle, |1\rangle\}$ are considered for each qubit i, where the conclusion does not hold. That is,

$$\langle b_3|\langle b_4|\ldots\langle b_N|\psi\rangle = |\alpha\rangle|\beta\rangle, \tag{1}$$

where the projections are carried out on the qubits $3,\ldots,N$ and the qubits one and two remain in the product state $|\alpha\rangle|\beta\rangle$ for any possible choice of the $\langle b_3|\langle b_4|\ldots\langle b_N|$. The $\langle b_i|$ can take the values 0 or 1. So, the product vector will in general depend on this choice and it is appropriate to write this dependency as

$$|\alpha\rangle = |\alpha(b_3,\ldots,b_N)\rangle \quad \text{and} \quad |\beta\rangle = |\beta(b_3,\ldots,b_N)\rangle. \tag{2}$$

What happens if the value of b_3 changes? The proof in Ref. [1] argues convincingly that then not *both* of the $|\alpha\rangle$ and $|\beta\rangle$ can

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change: If this were the case, a projection onto the superposition $\langle c_3| = \langle b_3 = 0| + \langle b_3 = 1|$, while keeping $\langle b_4| \dots \langle b_N|$ constant projects the system on the first two qubits in an entangled state. So, we have either

$$|\alpha\rangle = |\alpha(\circ, b_4, \dots, b_N)\rangle \quad \text{or} \quad |\beta\rangle = |\beta(\circ, b_4, \dots, b_N)\rangle, \tag{3}$$

where the "$\circ$" indicates that $|\alpha\rangle$ or $|\beta\rangle$ *for the given values of* $b_4, \dots, b_N$ does not depend on b_3.

The original proof continues the argument as follows: *Repeating the argument for other subspaces, we conclude that ... each index [b_i] actually appears in either $|\alpha\rangle$ or in $|\beta\rangle$ but not in both.* This conclusion is unwarranted. The point is that for a given set of $b_4, \dots, b_N$ one of the vectors (say, $|\alpha\rangle$ for definiteness) does not depend on b_3, but for another choice of $b_4, \dots, b_N$ the other vector $|\beta\rangle$ may be independent on b_3, while $|\alpha\rangle$ may depend on it. So, one cannot conclude that one of the vectors is *generally* independent.

The problem is best illustrated with a counterexample. Consider the four-qubit state

$$|\psi\rangle = \frac{1}{2}(|0000\rangle + |0101\rangle + |0110\rangle + |1111\rangle). \tag{4}$$

One can easily check that this is not a product state for any bipartition, so the state is genuine multiparticle entangled. Also, any projection into the computational basis on the particles three and four leaves the first two particles in a product state. We have for the dependencies:

$$|\alpha(00)\rangle = |0\rangle, \quad |\alpha(01)\rangle = |0\rangle, \quad |\alpha(10)\rangle = |0\rangle, \quad |\alpha(11)\rangle = |1\rangle, \tag{5}$$

and

$$|\beta(00)\rangle = |0\rangle, \quad |\beta(01)\rangle = |1\rangle, \quad |\beta(10)\rangle = |1\rangle, \quad |\beta(11)\rangle = |1\rangle, \tag{6}$$

so neither of these vectors does depend on a single index only.

Of course, if one chooses measurements in other directions on the qubits three and four, that is, one measures vectors like

$$|c_3\rangle = \cos(\gamma)|0\rangle + \sin(\gamma)|1\rangle \text{ and } |c_4\rangle = \cos(\delta)|0\rangle + \sin(\delta)|1\rangle, \tag{7}$$

then the remaining state on the qubits one and two is entangled. So the state $|\psi\rangle$ is not a counterexample to the main statement of the Lemma, but it demonstrates that proof requires some extra work.

Finally, if one accepts the step that each index $[b_i]$ occurs only in $|\alpha\rangle$ or $|\beta\rangle$, but not in both, one can conclude as demonstrated in Ref. [1] that the original state has to factorize, so it is not entangled.

3. Completing the argument

The previous section demonstrated that the proof of the Lemma in Ref. [1] is missing some discussions in order to be complete. In this section we provide a way to add the missing part. We prove the following statement:

Let $\mathbf{b}' = (b_3, b_4, \dots, b_N)$ with $b_i \in \{0, 1\}$ be the basis vectors which are used for the projection on the qubits $3, \dots, N$ and denote the remaining product state on the first two qubits by $|\alpha(\mathbf{b}')\rangle|\beta(\mathbf{b}')\rangle$. Then, $|\alpha(\cdot)\rangle$ depends only on some subset of the indices $\mathbf{b}'$, while $|\beta(\cdot)\rangle$ depends on the complement subset. This statement implies the correctness of the Lemma in Ref. [1].

The proof is done by assuming the opposite and reaching a contradiction. The opposite claim is that there exists an index i (without the loss of generality, we can take $i = 3$) and two sets of values for the remaining indices

$$\mathbf{b} = b_4, b_5, \dots, b_N \quad \text{and} \quad \mathbf{B} = B_4, B_5, \dots, B_N, \tag{8}$$

such that

$$|\alpha(0, \mathbf{b})\rangle \neq |\alpha(1, \mathbf{b})\rangle \quad \text{and} \quad |\beta(0, \mathbf{B})\rangle \neq |\beta(1, \mathbf{B})\rangle, \tag{9}$$

meaning that both depend on b_3. Here, $|\alpha(0, \mathbf{b})\rangle$ is a short-hand notation for $|\alpha(b_3 = 0, \mathbf{b})\rangle$. Also, the inequality symbol here and in the following indicates linear independence, i.e., $|\alpha(0, \mathbf{b})\rangle \neq \lambda|\alpha(1, \mathbf{b})\rangle$ for any $\lambda \neq 0$.

The vectors $\mathbf{b}$ and $\mathbf{B}$ differ in some entries, but in some entries they match. Without loss of generality, we can assume that they differ in the first k entries while the others are the same and equal to zero. More specifically, they can be taken of the form:

$$\mathbf{b} = 0\,0\,0\,\dots\,0\;\,0\,0\,\dots\,0,$$
$$\mathbf{B} = \underbrace{1\,1\,1\,\dots\,1}_{k}\;\underbrace{0\,0\,\dots\,0}_{N-k-3}. \tag{10}$$

Then the proof proceeds via induction on k. The precise statement we want to prove for all k is the following: Let the vectors $\mathbf{b}$ and $\mathbf{B}$ differ by at most at k terms. Then, if $|\alpha(0\mathbf{b})\rangle \neq |\alpha(1\mathbf{b})\rangle$, the equality $|\beta(0\mathbf{B})\rangle = |\beta(1\mathbf{B})\rangle$ must hold. The crucial point here is that on each induction step we need to use the already derived linear dependencies and independencies from all the previous induction steps, i.e. for all $k' < k$. We give the first ($k = 0 \mapsto k = 1$) and the second ($k = 1 \mapsto k = 2$) step of the induction explicitly, as this is needed in order to get the idea for the general case. This general case is discussed afterwards.

(a) The base case: If $k = 0$, then $\mathbf{b} = \mathbf{B}$ and the proof for this particular case is included in the discussion in Section 2 and in Ref. [1].

(b) $k = 0 \mapsto k = 1$:
As $|\alpha(\cdot)\rangle$ and $|\beta(\cdot)\rangle$ depend on b_3 and b_4 only, we can suppress the other indices and we write $|\alpha(00)\rangle, |\alpha(01)\rangle$, etc. Using this notation the problem boils down to showing that

$$|\alpha(00)\rangle \neq |\alpha(10)\rangle \quad \text{and} \quad |\beta(01)\rangle \neq |\beta(11)\rangle \tag{11}$$

cannot happen simultaneously. We show that if this would be true, it would contradict the assumption that the state after projection is a product state.
From step (a) we already know that the statement is correct for $k = 0$, which means that if only one value changes, then only one of $|\alpha(\cdot)\rangle$ and $|\beta(\cdot)\rangle$ can change. We can use this in the following way: For $x = 0$ or 1, if $|\alpha(0x)\rangle \neq |\alpha(1x)\rangle$, it follows that $|\beta(0x)\rangle = |\beta(1x)\rangle$. Furthermore, $|\alpha(x0)\rangle \neq |\alpha(x1)\rangle$ implies that $|\beta(x0)\rangle = |\beta(x1)\rangle$. And similarly, we can conclude equalities for the $|\alpha(\cdot)\rangle$ from inequalities of the $|\beta(\cdot)\rangle$.
Assuming that the statement in Eq. (11) can be satisfied, we would like to reach a contradiction. From the conditions in Eq. (11) and our previous argumentation it follows that

$$|\alpha(01)\rangle = |\alpha(11)\rangle \quad \text{and} \quad |\beta(00)\rangle = |\beta(10)\rangle. \tag{12}$$

Now there are two cases to be considered and in both cases a contradiction is reached:

1. The case $|\alpha(00)\rangle \neq |\alpha(01)\rangle$:
   Then from the result for $k = 0$ an equality follows for the $|\beta(\cdot)\rangle$, namely $|\beta(00)\rangle = |\beta(01)\rangle$. This implies with Eqs. (12) and (11) that $|\beta(10)\rangle \neq |\beta(11)\rangle$. Consequently, we get that $|\alpha(10)\rangle = |\alpha(11)\rangle$ and from Eq. (12) it follows that $|\alpha(10)\rangle = |\alpha(01)\rangle$.
   To sum up all the relations for $|\alpha(\cdot)\rangle$ and $|\beta(\cdot)\rangle$, we can write:

$$|\tilde{\alpha}\rangle \equiv |\alpha(01)\rangle = |\alpha(10)\rangle = |\alpha(11)\rangle \neq |\alpha(00)\rangle,$$
$$|\tilde{\beta}\rangle \equiv |\beta(00)\rangle = |\beta(01)\rangle = |\beta(10)\rangle \neq |\beta(11)\rangle. \tag{13}$$