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Parameter study of global and cluster synchronization in arrays of dry friction oscillators

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ABSTRACT

We investigate synchronization thresholds in arrays of identical classic stick-slip dry friction oscillators connected in a nearest neighbor fashion in closed and open ring network. Friction force is modeled by smoothened Stribeck model. Arrays of different length are checked in two parameter space (i.e., coupling coefficient vs. excitation frequency) for complete synchronization as well as cluster synchronization. Synchronization thresholds obtained by brute force numerical integration are compared with possible synchronization regions using the concept called master stability function in the form of two-oscillator reference probe. The results show existence of both complete synchronization and cluster synchronization regions in the investigated systems and confirm that two-oscillator probe can be applied for prediction of synchronization thresholds in systems with stick-slip phenomenon.

1. Introduction

Synchronous behavior of many dynamical systems has been observed for a long time. One of the first scientific record on synchronization was reported by Christiaan Huygens in the 17th century [1], when he observed mutual synchronization in two pendula hanging from a beam. He correctly concluded that small movements of the beam are responsible for the energy transfer between oscillating pendula, yielding to mutual synchronization. Nowadays, examples of synchronization can be encountered in various fields of science, such as mathematics, physics, engineering, biology, sociology [2–8]. Pikovsky et al. in their classic book on synchronization [9] define it as adjustments of rhythms of oscillating objects due to weak interaction. Recently, researchers have shown an increased interest in synchronization in nonlinear systems [10–14]. Different concepts and ideas in the area of synchronization have emerged (e.g. [15,16]).

The main goal of the investigation of synchronizability of a dynamical system is to find the synchronization thresholds, i.e., the strength of the coupling for which the synchronization occurs. In case of complete synchronization (CS), defined by Pecora and Carroll [17] as a situation when two trajectories converge in phase space, for identical oscillators (i.e., defined by the same ODE and parameters) one can use technique of master stability function (MSF) [18] or its simplified version for diagonal coupling [19] to establish the synchronization thresholds. To obtain the MSF using classic method one needs eigenvalues of connectivity matrix and Lyapunov exponents. For mechanical systems with discontinuities (e.g. friction oscillators), when it is difficult to calculate Lyapunov exponents, one can use two-oscillator probe, to evaluate synchronization thresholds [20]. This probe can be treated as a direct equivalent of the MSF when the so called real coupling between the oscillators is applied [21]. In practice, for mechanical systems only such a kind of coupling is possible.

In addition to the CS, one can also distinguish imperfect complete synchronization (ICS), when the oscillators almost synchronize (i.e., distance between oscillators in phase space converges to some small value). If a network consists of $N > 2$ identical oscillators one may distinguish at least two subsets, in which members are in sync with each other and out of sync with the members of other subsets. Such a subset is called cluster [22]. The topic of cluster synchronization has been addressed in many papers including [23–26]. Chaotic synchronization in networks of nearest neighbor topologies is investigated in [27], where the authors study 3-D cellular neural networks. An occurrence of synchronization windows in the parameter space is described in [28,29], where it is named as ragged synchronizability.

Friction force, as one of the ubiquitous forces in mechanics, is a result of the resistance of contacting surface to motion. In order to describe nature of friction, many models have been developed, considering various properties of friction phenomenon. These include classic Coulomb model [30], Stribeck effect [31], Lu-Gre model [32] and many others [33–39]. The choice of the

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appropriate model depends on dynamics between the interfaces. Application of nonlinear models is advised for systems with complex dynamics, while basic Coulomb model should be enough for statical problems. The occurrence of self-induced vibration (i.e., stick-slip) is characteristic for friction oscillators. The two contacting surfaces stick to each other and then slip in alternate manner. Stick-slip is the cause of the squeaking sound, one can hear from various mechanisms and the physical principle of the music produced by violins [40].

The classical stick-slip system consists of a drive (e.g. conveyor belt), elastic element (spring) and the mass which is pushed horizontally on the surface. Stick-slip friction oscillator is an example of a system with discontinuity, which means that the right hand sides of the differential equations are piecewise differentiable. The discontinuity originates from the change of the direction of friction force near zero relative tangential velocity between contacting surfaces, which is realized by means of signum function. This, however, requires special handling by the numerical routines. One of the solutions to that problem, encountered in the literature [41–43], is to smooth the model by using various sigmoid approximation of signum function (more details in Subsection 2.3). Understanding the stick-slip phenomenon is important in engineering, as it is crucial to model the behavior of mechanical elements [44–46].

Stick-slip phenomenon has been reported to accompany earthquakes [47]. Networks of coupled stick-slip oscillators, similar to the one used in this paper, have been adopted in seismology as a simplified model of earthquakes, known as Burridge–Knopoff model [48]. In this concept the blocks connected by springs on a rough surface mimic the behavior of continental plates. Different variants of Burridge–Knopoff model have been studied in [49–54]. Coupled friction oscillators are also discussed in [55–57], whereas [58] is focused on synchronization properties of such systems. Authors of [59] report an analogy between array of coupled friction oscillators and the dynamics of stock indexes.

In this paper we continue research, which was started in [21], where we checked arrays of self-induced friction oscillators only for the occurrence of complete synchronization. Our goal is to investigate the synchronization thresholds for open and closed rings of stick-slip friction oscillators in two-parameter space, for different length of arrays. We use the above mentioned two-oscillator probe to verify the findings of numerical calculations for the complete synchronization. Additionally, we aim to check the cluster synchronizability of the systems in question. The article is organized as follows. The methodology, including basic definitions, tools and mathematical model, is presented in Section 2. The results of numerical study are presented in Section 3. Finally, conclusions are drawn in Section 4.

2. Methodology

In this Section we introduce the reader to the definitions of synchronization used in the paper (Subsection 2.1) as well as the concept of master stability function and two-oscillator probe (Subsection 2.2), along with mathematical model of the system in question (Subsection 2.3).

2.1. Synchronization types

Complete synchronization is defined by Pecora and Caroll [17], as a situation when two trajectories converge in phase space, which giving two trajectories $\mathbf{x}(t)$ and $\mathbf{y}(t)$ can be defined as follows:

Definition 1. Complete synchronization of two dynamical system represented with their phase plane trajectories $\mathbf{x}(t)$, $\mathbf{y}(t)$, receptively, takes place when the following relation is fulfilled [60]:

$$\lim_{t\to\infty} \|\mathbf{x}(t) - \mathbf{y}(t)\| = 0. \tag{1}$$

Eq. (1) means that both oscillators behave exactly in the same manner (coincidence of phases and frequencies). Complete synchronization is possible only for identical oscillators (i.e., same ODEs and parameters). However, in reality due to parameters mismatch the oscillators may be almost in synchronization, with phase space separation between respective trajectories bellow small parameter ε.

Definition 2. The imperfect complete synchronization (ICS) of two dynamical systems represented with their phase plane trajectories $\mathbf{x}(t)$ and $\mathbf{y}(t)$ respectively, occurs when the following inequality is fulfilled [60]:

$$\lim_{t\to\infty} \|\mathbf{x}(t) - \mathbf{y}(t)\| \le \varepsilon, \tag{2}$$

provided ε is a small parameter.

Supposing the system consists of $N > 2$ identical oscillators one may distinguish one or more subsets for which the particular oscillators are in sync with each other and out of sync with the members of the other subsets. The subset of synchronized oscillators are called clusters. The motion of different clusters may be uncorrelated or one can observe a shift phase between them.

2.2. Master stability function and two-oscillator probe

As mentioned in the Introduction, MSF is used to determine synchronization thresholds for identical oscillators coupled in different configurations. In MSF the synchronizability of the network of identical coupled oscillators can be described by the eigenvalue spectrum of the connectivity matrix.

Let us couple N identical oscillators, which can be written in a block form as:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + (\sigma\mathbf{G} \otimes \mathbf{H})\,\mathbf{x}, \tag{3}$$

where $\mathbf{x} = (\mathbf{x_1}, \ldots, \mathbf{x_N}) \in \mathbb{R}^m$, $\mathbf{F}(\mathbf{x}) = (\mathbf{f}(\mathbf{x_1}), \ldots, \mathbf{f}(\mathbf{x_N}))$, σ – coupling coefficient defining the strength of the coupling, $\mathbf{G}$ is the connectivity matrix, which is Laplacian matrix describing the topology of connections between the nodes of the network, $\otimes$ is a Kronecker product of two matrices and finally $\mathbf{H}: \mathbb{R}^m \to \mathbb{R}^m$ output function of each oscillator's variables used in the coupling (identical for each node).

Derivation of variational equation of the network described in Eq. (3) yields to

$$\dot{\xi} = [\mathbf{I} \otimes D\mathbf{F} + \sigma\,(\mathbf{G} \otimes D\mathbf{H})]\,\xi, \tag{4}$$

where $\xi = (\xi_1, \ldots, \xi_N)$ – collection of perturbations, $\mathbf{I}$ – identity matrix, $D\mathbf{F}$, $D\mathbf{H}$ – respective Jacobians of system and output functions. The block diagonalization of Eq. (4) results in

$$\dot{\xi}_k = [D\mathbf{f} + \sigma\gamma_k D\mathbf{H}]\,\xi_k, \tag{5}$$

where ξ_k denotes different transverse modes of a perturbation from the synchronous state, γ_k stands for k-th eigenvalue of the connectivity matrix $\mathbf{G}$, $k = 0, 1, \ldots, N-1$. For $k = 0$ the eigenvalue is $\gamma_0 = 0$, thus reducing the Eq. (5) to variational equation of the separated node of the system

$$\dot{\xi}_k = D\mathbf{f}\xi_k, \tag{6}$$

which corresponds to the longitudinal direction located within the synchronization manifold. The other k-th eigenvalues correspond