Physics Letters A ••• (••••) •••–•••

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Equation of motion for estimation fidelity of monitored oscillating qubits ☆

H. Bassa [a], T. Konrad [a,c,*], L. Diósi [b], H. Uys [d,e,**]

[a] School of Chemistry and Physics, University of KwaZulu, Natal, Durban, South Africa
[b] Wigner Research Centre for Physics, Institute for Particle and Nuclear Physics, H-1525 Budapest 114, P.O.B. 49, Hungary
[c] National Institute of Theoretical Physics, Durban Node, School of Physics, University of KwaZulu, Natal, Durban, South Africa
[d] National Laser Centre, Council for Scientific and Industrial Research, Pretoria, Brummeria, 0184, South Africa
[e] Department of Physics, Stellenbosch University, Merriman Avenue, Stellenbosch, 7602, South Africa

ARTICLE INFO

Article history:
Received 27 February 2017
Received in revised form 4 May 2017
Accepted 8 May 2017
Available online xxxx
Communicated by P.R. Holland

Keywords:
Quantum state estimation
Unsharp measurement
Weak measurement
Quantum measurement
Qubits

ABSTRACT

We study the convergence properties of state estimates of an oscillating qubit being monitored by a sequence of *discrete*, unsharp measurements. Our method derives a differential equation determining the evolution of the estimation fidelity from a single incremental step. In case the oscillation frequency Ω is precisely known, the estimation fidelity converges exponentially fast to unity. For imprecise knowledge of Ω we derive the asymptotic estimation fidelity.

1. Introduction

High fidelity quantum state estimation is a key requirement in innumerable quantum control applications of quantum information processing, quantum simulation, quantum metrology, and quantum communication. Quantum state estimation [1–4] based on continuous or sequential unsharp (sometimes called weak) measurement [5–10] has opened new avenues for quantum control that obviate the need for repeated state preparation to execute tomography and allow, for example, real-time quantum closed-loop feedback. These principles have been brought to bear in different experimental platforms including microwave cavities [11] and superconducting qubits [12]. Improved sophistication in the unsharp measurement control toolbox [13] promises significant expansion beyond traditional open loop quantum control applications.

To achieve high fidelity control based on unsharp measurement the experimenter is forced to balance the benefit of allowing coherent dynamics to proceed subject to only weak perturbations, with the price of reduced information gain per measurement. As such, finding optimal estimation and control strategies are of prime importance. To make headway, detailed analytical descriptions of the measurement and estimation process are desirable.

In this paper we study the dynamics of state estimation fidelity during a sequence of discrete, unsharp measurements. Detailed analytical results are natural in the domain of *continuous* unsharp measurement, but we attempt here to place on a firmer footing the understanding of estimation dynamics during *sequential, discrete* measurements, as is natural in many experimental settings like trapped ions or microwave cavities.

We consider an estimation protocol wherein a state estimate is propagated by a Hamiltonian presumed to drive a laboratory quantum system which is also subject to sequential unsharp measurement. The state estimate is sequentially updated based on the outcome of

☆ This document is a collaborative effort.

* Corresponding author at: School of Chemistry and Physics, University of KwaZulu, Natal, Durban, South Africa.
** Principal corresponding author at: Department of Physics, Stellenbosch University, Merriman Avenue, Stellenbosch, 7602, South Africa.
E-mail addresses: konradt@ukzn.ac.za (T. Konrad), huys@csir.co.za (H. Uys).

http://dx.doi.org/10.1016/j.physleta.2017.05.013
0375-9601/© 2017 Published by Elsevier B.V.

each measurement on the actual quantum state with the same propagator as the system. Numerical simulations have demonstrated the convergence of the state estimate when all parameters in the Hamiltonian are precisely known [14,4] and even for the case of process tomography, treating the Hamiltonian parameters as state variables of a hybrid system [15–17].

Here we study the convergence of the state estimate for sequential unsharp measurements analytically, for the case of a two-level system undergoing Rabi oscillations. We start our study in Section 2 with a brief explanation of the state estimation protocol. In Section 3 we investigate the dynamics of the estimation fidelity, when the Hamiltonian is precisely known, or known within a specified error margin. This allows us to place bounds on the parameter space that grants high-fidelity state estimation.

2. State estimation and fidelity

A commonly used distance measure for the "closeness" of two quantum states is the estimation fidelity. It is defined as $F(\rho,\rho_e) = \mathrm{Tr}\left[\sqrt{\rho^{\frac{1}{2}}\rho_e\rho^{\frac{1}{2}}}\right]^2$, and is sometimes referred to as the *squared fidelity*. Here ρ and ρ_e are the density matrices describing the actual quantum state and the estimate of that quantum state, respectively. When both states are pure, i.e. $\rho = |\psi\rangle\langle\psi|$ and $\rho_e = |\psi^e\rangle\langle\psi^e|$, the fidelity takes the simple form $F(\psi,\psi^e) = |\langle\psi|\psi^e\rangle|^2$ [18]. The greater the fidelity, the more similar the two states are. It is 0 if, and only if, $|\psi\rangle$ and $|\psi^e\rangle$ are orthogonal pure states and it is 1 if, and only if, $|\psi\rangle = |\psi^e\rangle$.

Here we assume that $|\psi\rangle$ and $|\psi^e\rangle$ represent an *actual* and an *estimated* state, respectively. These states do not in general initially coincide, and our task is to exploit unsharp measurement with the aim of forcing $|\psi^e\rangle$ to converge onto $|\psi\rangle$ in the presence of ongoing dynamics. A single elementary step of the estimation protocol consists of a unitary time evolution of both states followed by an unsharp measurement (which is a probabilistic "filtering" operation) on the actual state. The estimated state is updated based on the result of the measurement on the actual system. Under appropriate circumstances successive repetitions of this elementary step lead to a faithful estimate of the actual state in real time.

More concretely, an elementary step of the method can be formulated mathematically as follows. First, the actual state evolves according to the Hamiltonian dynamics of the system specified by the evolution operator $\hat{U}(\Omega)$:

$$|\psi\rangle \Rightarrow \hat{U}(\Omega)|\psi\rangle \equiv |\psi'\rangle. \tag{1}$$

If the Hamiltonian is precisely known then the estimate is propagated using the same evolution operator. However, a (classical) parameter Ω specified in the Hamiltonian, such as the Rabi frequency in the case of Rabi oscillations, may be detuned away from the actual parameter. The estimated state is thus evolved using an estimated unitary operator $U(\Omega_e)$:

$$|\psi^e\rangle \Rightarrow \hat{U}(\Omega_e)|\psi^e\rangle \equiv |\psi^{e'}\rangle. \tag{2}$$

The task of estimating Ω_e has been the subject of earlier work [17]. The actual state then undergoes the following random change (collapse) under selective measurement,

$$|\psi'\rangle \Rightarrow \frac{1}{\sqrt{p_n}}\hat{M}_n|\psi'\rangle \equiv |\psi'_n\rangle \tag{3}$$

where $\hat{M}_n$ is the Kraus operator corresponding to the nth allowed measurement outcome, and $p_n = \langle\psi|\hat{M}_n^\dagger\hat{M}_n|\psi\rangle$ is the associated probability for the outcome. The Kraus operators are constrained via $\sum_n \hat{M}_n^\dagger\hat{M}_n = \sum_n \hat{E}_n = \mathbb{I}$, where we introduced the positive effects $\hat{E}_n = \hat{M}_n^\dagger\hat{M}_n$. The estimate $|\psi^e\rangle$ is updated using the outcome of the selective measurement just done on $|\psi'\rangle$:

$$|\psi^e\rangle \Rightarrow \frac{1}{\sqrt{p_n^e}}\hat{M}_n|\psi^{e'}\rangle \equiv |\psi_n^{e'}\rangle \tag{4}$$

with $p_n^e = \langle\psi^{e'}|\hat{E}_n|\psi^{e'}\rangle$. The divisor p_n^e is merely used to re-normalize the updated $|\psi^{e'}\rangle$; the statistics of the updates are determined by the probabilities p_n to observe outcome n.

We can now define the *average* change in fidelity after a single elementary step of the estimation protocol:

$$\Delta F = \sum_n p_n|\langle\psi'_n|\psi_n^{e'}\rangle|^2 \; - \; |\langle\psi|\psi^e\rangle|^2. \tag{5}$$

Using Eq. (3) and (4) we obtain

$$\Delta F = \sum_n \frac{|\langle\psi|\hat{U}^\dagger(\delta)\hat{E}'_n|\psi^e\rangle|^2}{\langle\psi^e|\hat{E}'_n|\psi^e\rangle} \; - \; |\langle\psi|\psi^e\rangle|^2, \tag{6}$$

where $\delta = \Omega - \Omega_e$ and $\hat{E}'_n = \hat{U}^\dagger(\Omega_e)\hat{E}_n\hat{U}(\Omega_e)$ are the time-varying effects.

As is clear from Ref. [2], in the case where the Hamiltonian is precisely known ($\delta = 0$) the above update follows the spirit of the classical Bayesian update and we expect intuitively that the measured actual $|\psi\rangle$ and the updated estimate $|\psi^e\rangle$ come "closer" to each other. The method performs suitably well in various quantum estimation situations [14,4,11,19]. In what follows, we will use Eq. (6) to derive time dependence of the estimation fidelity convergence of oscillating qubits when the Hamiltonian is precisely know ($\delta = 0$) and find the asymptotic estimation fidelity when the Hamiltonian is not precisely known ($\delta \neq 0$).

Please cite this article in press as: H. Bassa et al., Equation of motion for estimation fidelity of monitored oscillating qubits, Phys. Lett. A (2017), http://dx.doi.org/10.1016/j.physleta.2017.05.013