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Bounds on mixedness and entanglement of quantum teleportation resources

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ABSTRACT

For a standard teleportation protocol, rank dependent upper bounds on Von Neumann entropy and linear entropy beyond which the states of respective ranks cease to be useful for quantum teleportation are derived analytically for a general $d \times d$ bipartite systems. For two qubit mixed states, we obtain rank dependent lower bound on concurrence below which the state is useless for quantum teleportation. For two qubit system, we construct mixed states of different ranks that exhibit theoretical rank dependent upper bounds on the measures of mixedness and lower bounds on the concurrence.

1. Introduction

Entangled states are used as resource for quantum teleportation [1–4]. The success of quantum teleportation is quantified by teleportation fidelity. The strength of entanglement is proportional to the success of teleportation for pure entangled resources. Maximally entangled pure states give maximum teleportation fidelity of unity. On the other hand, for mixed states, success of teleportation depends, in addition to strength of entanglement, on mixedness of the state [5,6]. It is known that entanglement itself depends on mixedness of the state [8,7]. It was shown in [9] that, for mixed entangled states, there is an upper bound on mixedness above which the state is useless for quantum teleportation. Upper bounds on measures of mixedness, namely, Von Neumann entropy and linear entropy were obtained for a general bipartite $d \times d$ state. Mixed states can be classified according to their rank r, where the rank r varies between 2 and d^2 for a bipartite $d \times d$ system. In the present work, we obtain rank dependent upper bounds on measures of mixedness, above which the states of respective ranks become useless for quantum teleportation in a standard teleportation protocol.

In our previous work [5], we observed that mixedness and entanglement of a mixed state resource independently influence teleportation. For a state with a fixed value of mixedness, the state being entangled is only necessary for it to be a resource for quantum teleportation, but not sufficient. States with low mixedness and high entanglement turn out to be ideal resources for quantum teleportation. We argued based on the numerical work on a class of maximally entangled mixed states that there exists a rank dependent lower bound on a measure of entanglement such as concurrence, below which the states are useless for quantum teleportation. In this letter, we derive rank dependent lower bounds on concurrence for a bipartite 2×2 systems.

It is know that Werner state, defined as a probabilistic mixture of maximally entangled pure state and maximally mixed separable state, exhibits highest mixedness for a given teleportation fidelity among all the two qubit mixed entangled states known in the literature. A pair of Werner states [10] is used to teleport unknown entangled quantum states. Werner state [11] is a rank 4 state. We observe that Werner state exhibits theoretical upper and lower bounds on measures of mixedness and entanglement respectively for rank 4 states. We construct mixed states of lower ranks that exhibit the respective rank dependent bounds obtained on measures of mixedness and entanglement for a given value of teleportation fidelity.

2. Upper bounds on measures of mixedness

We consider two measures of mixedness of a state ρ, namely, von Neumann entropy defined as $S(\rho) = -Tr(\rho \ln \rho)$ [12] and linear entropy given as $S_L = \frac{d^2}{d^2-1}[1 - Tr(\rho^2)]$. Further, the maximum achievable teleportation fidelity of a bipartite $d \times d$ system

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in the standard teleportation [6] scheme is $F = \frac{fd+1}{d+1}$, where f is the singlet fraction of ρ given by $f(\rho) = max_{\psi}\langle\psi|\rho|\psi\rangle$. Here the maximum is over all maximally entangled states. And maximum fidelity achieved classically is $F_{cl} = \frac{2}{d+1}$. This shows that the state is useful for quantum teleportation if $f > \frac{1}{d}$ $(F(\rho) > F_{cl})$. Along the lines of successful studies where for two qubit system, $f > \frac{1}{2}$ is a sufficient condition for entanglement distillability of mixed states and their super activation [13] and is a condition for purification of a state using quantum privacy amplification procedure [14], we employ $f > \frac{1}{d}$ as a condition failing which the state is useless for quantum teleportation in standard teleportation protocol.

Let ρ_r denote a bipartite mixed state of $d \times d$ system whose rank is r, $2 \le r \le r_{max} = d^2$. Firstly, we prove a theorem that gives rank dependent upper bounds on von Neumann entropy, above which the state is useless for quantum teleportation.

Theorem. *If Von Neumann entropy $S(\rho_r)$ of a given state ρ_r of rank r of $d \times d$ system exceeds $\ln\sqrt{r_{max}} + (1 - \frac{1}{\sqrt{r_{max}}})\ln\frac{r-1}{\sqrt{r_{max}}-1}$, then the state is not useful for quantum teleportation.*

Proof. Consider ρ_r, a mixed state of rank r of $d \times d$ system $(2 \le r \le r_{max} = d^2)$. The singlet fraction $f(\rho_r) > \frac{1}{d}$ implies the state is useful for quantum teleportation. And it is known that Von Neumann entropy $S(\rho_r)$ of a given state ρ_r is less than or equal to Shannon entropy. If $\lambda_1, \lambda_2, \cdots, \lambda_r$ are eigenvalues of ρ_r in descending order, then the Shannon entropy is maximized for $\lambda_1 = f$ and other $r-1$ eigenvalues are equal. Subjected to the constraint $f > \frac{1}{\sqrt{r_{max}}}$, we get the upper bound as

$$S^{\star}(\rho_r) = \sum_{i=1}^{r}\lambda_i \ln\lambda_i = \frac{\ln\sqrt{r_{max}}}{\sqrt{r_{max}}} - \left(1 - \frac{1}{\sqrt{r_{max}}}\right)\ln\left(\frac{1}{r-1}\left(1 - \frac{1}{\sqrt{r_{max}}}\right)\right) \quad (2.1)$$

This implies,

$$S(\rho_r) \le \ln\sqrt{r_{max}} + \left(1 - \frac{1}{r_{max}}\right)\ln\frac{r-1}{\sqrt{r_{max}}-1} \quad (2.2)$$

This shows that for $S(\rho_r)$ violating eq. (2.2), the singlet fraction cannot be greater than $\frac{1}{\sqrt{r_{max}}}$. Thus, we prove if $S(\rho_r) > \ln\sqrt{r_{max}} + (1 - \frac{1}{r_{max}})\ln\frac{r-1}{\sqrt{r_{max}}-1}$, state ρ_r is useless for quantum teleportation in a standard teleportation protocol.

We also obtain an analytical expression for the rank dependent upper bound on linear entropy as a measure of mixedness as

$$S_L^{\star}(\rho_r) = \frac{r\,(r_{max}-1) - 2\sqrt{r_{max}}\left(\sqrt{r_{max}}-1\right)}{(r_{max}-1)\,(r-1)} \quad (2.3)$$

For $d = 2$, we have

$$S_L^{\star}(\rho_r) = \frac{3r-4}{3(r-1)} \quad (2.4)$$

where r is the rank of the state, which varies from 2 to 4. In our previous work [5], based on the analysis of a class of maximally entangled mixed states of 2×2 bipartite system given in [15], we observed that for a given value of linear entropy, there exists a rank dependent upper bound on the teleportation fidelity and the upper bound increases with increase in the rank. This is equivalent to stating that for a given value of optimal teleportation fidelity, there exists a rank dependent upper bound on linear entropy and the upper bound increases with rank. The above result allows us to calculate the upper bounds explicitly. If the teleportation fidelity is fixed as $2/3$ $(f = 1/2)$, the classical fidelity, the upper bound on linear entropy for states of second, third and fourth ranks are 2/3, 5/6 and 8/9 respectively. States with linear entropy above the respective rank dependent upper bounds are useless for quantum teleportation. Our results are in agreement with the observation that there are no entangled states if the linear entropy is greater than 8/9 [16].

A class of two qubit maximally entangled mixed states (MEMS) is constructed by Ishizaka et al. [15,17] and the construction is as follows.

$$\rho = \lambda_1|\psi^-\rangle\langle\psi^-| + \lambda_2|00\rangle\langle 00| + \lambda_3|\psi^+\rangle\langle\psi^+| + \lambda_4|11\rangle\langle 11| \quad (2.5)$$

where λ_i are the eigenvalues of the state ρ $(\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \lambda_4$ and $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1)$ and maximally entangled Bell states $|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$. The Werner state, which is a convex sum of maximally entangled pure state and maximally mixed separable state, given by

$$W_4 = (1-p)\frac{I_4}{4} + p|\psi^+\rangle\langle\psi^+| \quad (2.6)$$

corresponds to a choice of eigenvalues given by $\lambda_1 = \frac{1+3p}{4}$ and $\lambda_2 = \lambda_3 = \lambda_4 = \frac{1-p}{4}$. Werner state is a state of rank 4. The singlet fraction for Werner state is $\frac{1+3p}{4}$ and linear entropy is estimated as $1 - p^2$. The value of linear entropy at which fidelity is equal to classical limit [18] can be found as $\frac{8}{9}$, which is same as the upper bound obtained above for states of rank 4. This correspondence motivated us to construct lower rank maximally entangled mixed states that exhibit theoretical bounds derived in this section. A rank 3 boundary MEMS can be constructed by using eigenvalues as $\lambda_1 = \frac{1+2p}{3}$, $\lambda_2 = \lambda_3 = \frac{1-p}{3}$ and $\lambda_4 = 0$. We have $f(W_3) = \frac{1+2p}{3}$ and linear entropy is equal to $S_L(W_3) = \frac{8(1-p^2)}{9}$. Thus it is clear that $f(W_3) \le \frac{1}{2}$ for a value of linear entropy greater than $\frac{5}{6}$, which is the obtained upper bound beyond which a rank 3 state is useless for quantum teleportation. To construct a rank 2 boundary MEMS we take $\lambda_1 = \frac{1+p}{2}$, $\lambda_2 = \frac{1-p}{2}$ and $\lambda_3 = \lambda_4 = 0$. We get singlet fraction and linear entropy as $\frac{1+p}{2}$ and $\frac{2(1-p^2)}{3}$ respectively. It clearly shows that rank 2 boundary MEMS is useless for teleportation when linear entropy is greater than $\frac{2}{3}$, coinciding with the theoretical upper bound shown above. Thus, we illustrate that the constructed lower rank boundary MEMS exhibit the theoretical upper bounds on the linear entropy for a given value of optimal teleportation fidelity. □

3. Lower bounds on measure of entanglement for two qubit mixed states

The relationship between the concurrence as a measure of entanglement of state ρ and it's purity is well studied, the degree of entanglement decreases as purity decreases. The maximum possible value of concurrence of a state ρ for a spectrum of eigenvalues [15,17,20] $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ in descending order is given as

$$C_{max}(\rho) = max(0, \lambda_1 - \lambda_3 - 2\sqrt{\lambda_2\lambda_4}) \quad (3.1)$$

In our previous work [5], we also observed that there exist a rank dependent lower bound on concurrence for a fixed value of teleportation fidelity and the lower bound decreases with increase in the rank of the state. In the present work, we obtain the exact rank dependent lower bounds on the concurrence for a 2×2 bipartite states. For a given value of singlet fraction f, a two qubit state of rank r $(2 \le r \le 4)$ with eigenvalue $\lambda_1 = f$ and other $r-1$ eigenvalues equal leads to maximum mixedness. Since concurrence decreases with increase in mixedness, the maximum concurrence for a fixed value of mixedness or equivalently the eigenspectrum gives a lower bound on concurrence of all mixed entangled states of rank r.

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