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Delta shock waves in shallow water flow

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ABSTRACT

The shallow-water equations for two-dimensional hydrostatic flow over a bottom bathymetry $b(x)$ are

$$h_t + (uh)_x = 0,$$
$$u_t + \left(gh + u^2/2 + gb\right)_x = 0.$$

It is shown that the combination of discontinuous free-surface solutions and bottom step transitions naturally lead to singular solutions featuring Dirac delta distributions. These singular solutions feature a Rankine–Hugoniot deficit, and can readily be understood as generalized weak solutions in the variational context, such as defined in [14,23]. Complex-valued approximations which become real-valued in the distributional limit are shown to extend the range of possible singular solutions. The method of complex-valued weak asymptotics [23,24] is used to provide a firm link between the Rankine–Hugoniot deficit and the singular parts of the weak solutions. The interaction of a surface bore (traveling hydraulic jump) with a bottom step is studied, and admissible solutions are found.

1. Introduction

The standard theory of hyperbolic conservation laws in one spatial dimension can be applied to systems which are strictly hyperbolic and genuinely nonlinear. If initial data are given which have small enough total variation, then it can be shown that there is a solution which exists for all times [17,19,35]. This solution will in general be discontinuous, featuring a number of jumps. However, if one of the above hypotheses is not satisfied, the initial-value problem cannot in general be resolved (see e.g. [4–6,9,12,29,31,36]) and further restrictions on the data need to be introduced, such as for example in [36]. In fact, in some cases, even the Riemann problem cannot be solved.

Starting with the work reported on in [27], existence of solutions was shown to be possible if the space of solutions was extended to include Radon measures. In particular, such non-standard solutions were shown to contain Dirac δ-distributions attached to the location of certain discontinuities. As was shown in [26], the incorporation of such δ-shocks is equivalent to relaxing one or more of the required Rankine–Hugoniot conditions for classical shocks, and it may be shown that the strength of the Dirac δ-distribution associated to a certain shock is a precise measure of the deficit in the Rankine–Hugoniot conditions which are required to obtain a solution.

In the present work, we consider the shallow-water system, and show how δ-shocks arise naturally if this theory is to describe the physics of the underlying problem adequately. Indeed, unlike the situation from [24] where the δ-distribution was adjoined to the surface excursion, here we shall see that δ-naturally appears as part of the velocity as a measure of the Rankine–Hugoniot deficit. Alternative approaches for physical explanations of the appearance of delta functions and Rankine–Hugoniot deficits in the context of shallow-water dynamics were given in [15], where a localized jet is considered, and [21], where mixing closures for a two-layer system were proposed.

The shallow-water system describes the flow of an inviscid fluid in a long channel of small uniform width, and is used as a standard model in the field of hydraulics which is fundamental for example in the study of river bores and storm surges in rivers and channels [19,37]. If it can be assumed that the bottom is flat (such as in a laboratory situation), the system is usually written in the form

$$\partial_t h + \partial_x (uh) = 0, \text{ (mass conservation)}, \tag{1.1}$$
$$\partial_t (uh) + \partial_x \left(u^2 h + g\frac{h^2}{2}\right) = 0, \text{ (momentum balance)}, \tag{1.2}$$

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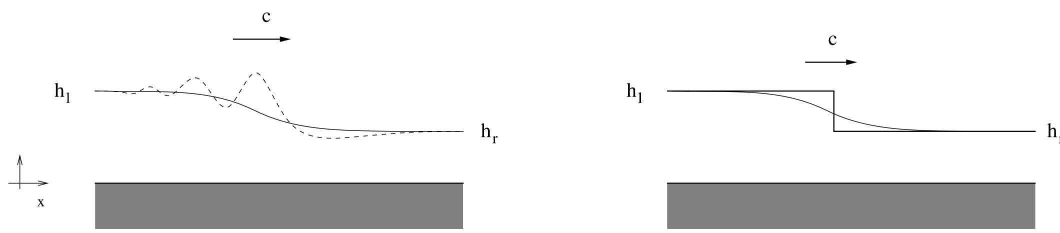

Fig. 1. Left panel: Surface profile of a traveling hydraulic jump (undular bore). Right panel: shallow-water approximation.

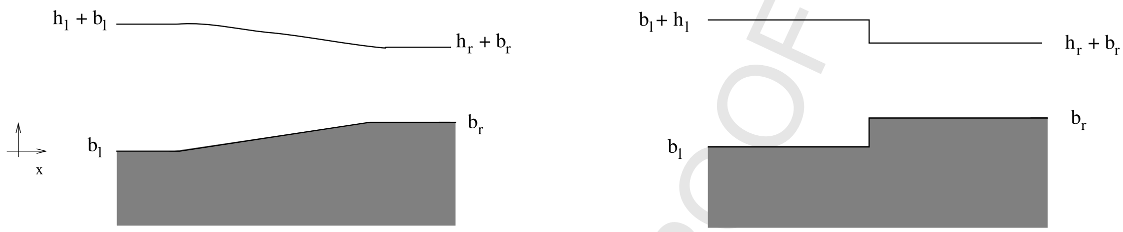

Fig. 2. Left panel: Surface profile over a bottom transition. Right panel: shallow-water approximation.

where h denotes the total flow depth, u represents an average horizontal velocity, and g is the gravitational constant. For smooth solutions, an equivalent system is

$$\partial_t h + \partial_x (uh) = 0, \tag{1.3}$$

$$\partial_t u + \partial_x\Big(\tfrac{u^2}{2} + gh\Big) = 0, \tag{1.4}$$

and it is immediately clear that mass and momentum conservation in discontinuous solutions lead to a Rankine–Hugoniot deficit in (1.4). One might conclude that it would therefore be best to avoid the system (1.3)–(1.4) in favor of the system (1.1)–(1.2). The theory for this system is well developed, and both the initial-value problem and the Riemann problem can be solved [19]. The conservation of energy is formulated as

$$\partial_t \left(h\tfrac{u^2}{2} + g\tfrac{h^2}{2}\right) + \partial_x \left(guh^2 + h\tfrac{u^3}{2}\right) = 0 \tag{1.5}$$

and this then serves as a mathematical entropy.

On the other hand, in many practical situations, the assumption of a flat bottom is too restrictive. If an uneven bed is introduced, the equations take the form

$$\partial_t h + \partial_x (uh) = 0 \quad \text{(mass conservation)} \tag{1.6}$$

$$\partial_t u + \partial_x \left(gh + \tfrac{u^2}{2}\right) = -gb_x \tag{1.7}$$

$$\partial_t (uh) + \partial_x \left(u^2 h + g\tfrac{h^2}{2}\right) = -ghb_x \quad \text{(momentum balance)} \tag{1.8}$$

$$\partial_t \left(h\tfrac{u^2}{2} + g\tfrac{h^2}{2} + bh\right) + \partial_x \left(guh(h+b) + h\tfrac{u^3}{2}\right) = 0 \quad \text{(energy balance)} \tag{1.9}$$

In this system, the function $b(x)$ measures the rise of the bed above a certain reference level at $z = 0$. The function $h(x,t)$ measures the flow depth of the fluid, so that $b(x) + h(x,t)$ measures the position of the free surface relative to the reference point $z = 0$ (see Fig. 1 and Fig. 2).

Again, for discontinuous solutions, mass and momentum conservation are to be satisfied, so that (1.7) and the energy equation (1.9) will feature a Rankine–Hugoniot deficit. In the case of a shock over a bottom step, momentum is not conserved because of the lateral pressure force appearing in (1.8), and in this case energy conservation needs to be specified. Therefore, in this case a Rankine–Hugoniot deficit will be introduced in (1.8).

In this paper we will address the relatively simple situation of a flow of a shock wave over a bottom step. The shock wave is governed by the Rankine–Hugoniot conditions originating from mass and momentum conservation, i.e. by (1.6) and (1.8). On the other hand, as explained above, a discontinuity over a bottom step is governed by the Rankine–Hugoniot conditions originating from mass and energy conservation, i.e. by (1.6) and (1.9). Thus it is plain that it is not possible to resolve the underlying physical problem with the use of only two governing equations. If the goal is to maintain the classical modeling approach of describing a situation with a certain fixed set of equations so that the number of equations and unknowns is the same, it is necessary to allow for Rankine–Hugoniot deficits and the corresponding incorporation of singular delta shocks.

Thus in order to salvage the classical modeling approach, we propose the following procedure. Use the system (1.6)–(1.7) as the system to be solved, and use the corresponding Rankine–Hugoniot conditions for momentum or energy conservation in the appropriate places. Since these can be made explicit via delta-shock waves, the system is self-sufficient. For further study, the system (1.6)–(1.7) can be cast in conservative form by writing

$$\left.\begin{aligned} \partial_t h + \partial_x(uh) &= 0, \\ \partial_t u + \partial_x \left(gh + \tfrac{u^2}{2} + gb\right) &= 0. \end{aligned}\right\} \tag{1.10}$$

The plan of the present paper is as follows. In Section 2, surface discontinuities over a flat bottom are studied, and it is shown that if these discontinuous solutions satisfy mass and momentum conservation, and the required energy loss, then the total head $\frac{1}{2g}u^2 + h$ cannot be conserved. Thus a Rankine–Hugoniot deficit is needed in the second equation in (1.10). The solution is verified both in the weak asymptotic context, and in the weak variational context. In Section 3, bottom step transitions are studied. In Section 4, the interaction of a discontinuous moving surface profile with a bottom step is investigated.

2. Surface discontinuities

In this section, we briefly review the theory surrounding discontinuous solutions of the shallow-water system, and we show that an admissible weak solution conserving mass and momentum, and dissipating mechanical energy must give rise to a Rankine–Hugoniot deficit for the conservation equation for the total head. Then, it is described how such a singular solution can be understood as a delta shock wave in the framework of the weak asymptotic method, and in the generalized variational framework.