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On Hamiltonian formulations of the $\mathcal{C}_1(m, a, b)$ equations

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ABSTRACT

In this letter we re-address a class of genuinely nonlinear third order dispersive equations; $\mathcal{C}_1(m, a, b)$: $u_t + (u^m)_x + \frac{1}{b}[u^a(u^b)_{xx}]_x = 0$, which among other solitary structures admit compactons, and demonstrate that certain subclasses of these equations may be cast into Hamiltonian and Lagrangian formulations resulting in new conservation laws, some of which are *nonlocal*. In particular, the new nonlocal conservation law of the $K(n, n)$ equations enables us to prove that the response to a certain class of excitations cannot contain only compactons.

1. Introduction

The $\mathcal{C}_N(m, a, b)$ equations

$$\mathcal{C}_N(m,a,b): \quad u_t + (u^m)_x + \frac{1}{b}[u^a \nabla^2 u^b]_x = 0,$$
$$m > 1, 1 < n \doteq a + b \tag{1.1}$$

where N denotes the spatial dimension, have been introduced in [1] as a prototype of dynamics shaped by a genuinely nonlinear dispersion with a number of its cases studied in [1,2]. In its N-dimensional form it may be seen as a generalization of the Zakharov–Kuznetsov equation [3]. In particular, in the $N = 1$ case we have

$$\mathcal{C}_1(m,a,b): \quad u_t + (u^m)_x + \frac{1}{b}[u^a (u^b)_{xx}]_x = 0, \tag{1.2}$$

which for $a = 0$ reduces to the more familiar $K(m, n)$ equations [4] (all coefficients may be normalized up to a sign)

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \quad 1 < m, n, \tag{1.3}$$

well known for supporting compactons [4,5].

Whereas from both physical and mathematical points of view the $\mathcal{C}_1(m, a, b)$ family of equations is a natural generalization of the celebrated KdV and mKdV equations (corresponding to $a = 0, b = 1$ and $m = 2, 3$ respectively in (1.2)), their variational structures do not extend in any intuitive way to the general case. The miracle of integrability is already lost in the semilinear case for $m > 3$. Yet, whereas the loss of integrability, being a very delicate feature, is to be expected, the variational structure is a basic physical feature of the underlying system. Thus its vanishing is a bit puzzling.

We shall show that certain subcases of Eqs. (1.3) and (1.2) admit variational formulations, some of which are quite different from the conventional ones, and consider the resulting implications. To this end one has to employ nonlocal representations which unify the various cases. We shall also apply certain mappings to subcases of $\mathcal{C}_1(m, a, b)$ provided that the image is a subcase of $\mathcal{C}_1(m, a, b)$. We start by recapping few of the known features of the $\mathcal{C}_1(m, a, b)$ equations.

1.1. Certain features of the $\mathcal{C}_1(m, a, b)$ equations

The $\mathcal{C}_1(m, a, b)$ equations admit *both traveling and stationary compactons*. Let

$\omega := b + 1 - a$.

ω plays a crucial role in shaping the dynamics; if $\omega > 0$, *the traveling compactons* are evolutionary (note that for $K(m, n)$, $\omega = n + 1$), whereas if $\omega < 0$ *stationary compactons* are evolutionary and emerge out of compact initial excitations [1,6]. Thus, for instance, whereas the $\mathcal{C}_1(2, 1, 1)$ equation ($\omega = 1$) supports an evolutionary traveling compacton

$$u(x,t) = 2\lambda \cos^2\left(\frac{x - \lambda t}{2}\right) H(\pi - |x - \lambda t|) \tag{1.4}$$

where $H(s) = \begin{cases} 1, & s \geq 0 \\ 0, & s < 0 \end{cases}$ is the Heaviside function, in the $\mathcal{C}_1(4, 3, 1)$ case ($\omega = -1$) the stationary compactons

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$$u(x,t) = u_0 \cos(x) H\left(\frac{\pi}{2} - |x|\right) \tag{1.5}$$

are the ones to emerge from a compact initial excitation. As to the regularity of the dispersive part at compacton's edge, we note that it can be rewritten as $\left[u^3 u_{xx}\right]_x = \left[\frac{1}{4}(u^4)_{xx} - 3u^2 u_x^2\right]_x$. It is thus clear that all terms are well defined at the singularity.

The $K(n,n), n > 1$ equations (which correspond to $\mathcal{C}_1(n,0,n)$), admit a finite number of local conservation laws [7]:

$$I_1 = \int u dx,\ I_\omega = \frac{1}{\omega}\int u^\omega dx,\ I_3 = \int u \sin(x) dx \text{ and}$$
$$I_4 = \int u \cos(x) dx. \tag{1.6}$$

I_1 and I_ω are also conserved in the general $\mathcal{C}_1(m,a,b)$ equations (1.2). In subsection 2.6 we shall briefly address the $m = 1, -\frac{1}{2}, -2$ cases which admit an infinite number of conservation laws, but do not support compactons.

2. Hamiltonian formulations and the main results

We shall now cast a number of subclasses of (1.2) as Hamiltonian evolution equations, but first we recall certain basic facts:

Definition 1. [8] An evolution equation $u_t = K(x, u, u_x, u_{xx}, \ldots)$ is said to be *Hamiltonian* if it can be written in the form

$$u_t = \mathcal{D} \cdot \frac{\delta}{\delta u} H[u],$$

where $H[u]$ is a functional and $\mathcal{D}$ a Hamiltonian operator – a skew symmetric operator which satisfies the Jacobi identity.

For later use we record a number of well known Hamiltonian operators, [8,9]:

1. All skew symmetric operators $\mathcal{D}$ with coefficients that do not depend on u and its derivatives.
2. $\mathcal{D} = \pm D_x^3 + 2(\lambda u + k(x))D_x + (\lambda u_x + k'(x)),\ \lambda \in \mathbb{R}, k(x) \in C^1(\mathbb{R})$.
3. $\mathcal{D} = \pm D_x^3 + \lambda D_x \cdot u D_x^{-1} \cdot u D_x$.

Clearly, the $K(n,n)$ equations may be cast into a Hamiltonian form

$$u_t = \mathcal{D}\frac{\delta}{\delta u} I_\omega(u), \quad \mathcal{D} = -D_x - D_x^3. \tag{2.1}$$

Definition 2. Let $Q = Q(x, u, u_x, u_{xx}, ..)$. Its Fréchet derivative is the operator D_Q defined by

$$D_Q = \sum_k \frac{\partial Q}{\partial u_k} \cdot D_x^k.$$

Its adjoint $D_Q^\star$ is

$$D_Q^\star = \sum_k (-D_x)^k \cdot \frac{\partial Q}{\partial u_k}.$$

We shall utilize the following two Theorems:

Theorem 1. [10] *Let $\mathcal{D}$ be a Hamiltonian operator depending on $(x, u, u_x, u_{xx}, \ldots)$ and let $y = P(x, u, u_x, u_{xx}, \ldots)$, $w = Q(x, u, u_x, u_{xx}, \ldots)$ be related to (x, u) by a differential substitution. Then the corresponding Hamiltonian operator in the (y, w) variables is*

$$\tilde{\mathcal{D}} = (D_x P)^{-1} \boldsymbol{J} \cdot \mathcal{D} \cdot \boldsymbol{J}^\star$$

where the operator $\boldsymbol{J}$ is given by

$$\boldsymbol{J} = D_x P \cdot D_Q - D_x Q \cdot D_P$$

and $\boldsymbol{J}^\star$ is its adjoint

$$\boldsymbol{J}^\star = D_Q^\star \cdot D_x P - D_P^\star \cdot D_Q.$$

Theorem 2. [11] *The Hamiltonian evolution equation $u_t = D_x \frac{\delta}{\delta u} H$ is equivalent to the Euler–Lagrange equation for the variational problem*

$$\mathcal{L} = \int \left(\int \frac{1}{2} \psi_x \psi_t dx - H[\psi_x] \right) dt$$

where $\psi(x,t)$ is the potential function of $u(x,t)$, so that $\psi_x = u$.

Introducing a change of variables which maps our equations into $u_t = D_x \frac{\delta}{\delta u} H$, we shall use Theorem 2 in the following subsections to derive a Lagrangian associated with a few subcases of $\mathcal{C}_1(m,a,b)$.

2.1. The $K(n,n)$ equations

We start with

Lemma 1. *The $K(n,n)$ equations admit a Lagrangian formulation and the nonlocal conservation law*

$$\frac{d}{dt}\int \left[u(1 + D_x^2)^{-1} u \right] dx = 0.$$

Proof. Referring to the Hamiltonian operator $\mathcal{D} = -D_x - D_x^3$, we apply the change of variables

$$v = Q(x, u, u_x, u_{xx}, \ldots) = L^{-1} u$$

where L is the pseudo-differential operator

$$L = \sqrt{D_x^2 + 1} = D_x^1 + \frac{1}{2} D_x^{-1} - \frac{1}{8} D_x^{-3} \ldots$$

to the $K(n,n)$ equations. Then, by Theorem 1 in (x, v) variables, the Hamiltonian operator is

$$\tilde{\mathcal{D}} = D_Q \cdot \mathcal{D} \cdot D_Q^\star = L^{-1} \cdot (-D_x - D_x^3) \cdot (-L^{-1}) = D_x.$$

Thus, in (x, v) variables equation (1.3), $m = n$ reads

$$v_t = D_x \frac{\delta}{\delta v} \int \frac{1}{n+1} (L(v))^n dx. \tag{2.2}$$

By Theorem 2, the Lagrangian of (2.2) is given by:

$$\mathcal{L} = \iint \left[\frac{1}{2} \psi_x \psi_t - \frac{1}{n+1} (L D_x \psi)^{n+1} \right] dx dt \tag{2.3}$$

where $\psi_x = v$. Using Noether relations between symmetries and conservation laws, invariance under space translation implies conservation of $\int v^2 dx$ which in the original variables is a non-local conservation law

$$\frac{d}{dt} I_2(u) = 0, \quad I_2(u) = \int u L^{-2}[u] dx. \tag{2.4}$$

The functionals I_1, I_2, I_ω may then be looked upon as a generalized mass, a generalized momentum (invariance under space translation) and a generalized energy (invariance under time translation), respectively, for the $K(n,n)$ equations. □

Note that the new conservation law could have been deduced directly from the Hamiltonian, but from a mathematico-physical