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Stability of soliton families in nonlinear Schrödinger equations with non-parity-time-symmetric complex potentials

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ABSTRACT

Stability of soliton families in one-dimensional nonlinear Schrödinger equations with non-parity-time ($\mathcal{PT}$)-symmetric complex potentials is investigated numerically. It is shown that these solitons can be linearly stable in a wide range of parameter values both below and above phase transition. In addition, a pseudo-Hamiltonian–Hopf bifurcation is revealed, where pairs of purely-imaginary eigenvalues in the linear-stability spectra of solitons collide and bifurcate off the imaginary axis, creating oscillatory instability, which resembles Hamiltonian–Hopf bifurcations of solitons in Hamiltonian systems even though the present system is dissipative and non-Hamiltonian. The most important numerical finding is that, eigenvalues of linear-stability operators of these solitons appear in quartets $(\lambda, -\lambda, \lambda^*, -\lambda^*)$, similar to conservative systems and $\mathcal{PT}$-symmetric systems. This quartet eigenvalue symmetry is very surprising for non-$\mathcal{PT}$-symmetric systems, and it has far-reaching consequences on the stability behaviors of solitons.

1. Introduction

Parity-time ($\mathcal{PT}$) symmetry is currently at the forefront of research in physics and applied mathematics (see [1,2] for reviews). This concept started out in quantum mechanics, where it was observed that complex potentials with parity-time symmetry could still exhibit all-real spectra even though the underlying Schrödinger operator is non-Hermitian [3]. Later, this concept spread to optics, where it was realized that optical waveguides with even refractive-index profiles and odd gain-loss distributions constitute $\mathcal{PT}$-symmetric systems [4]. In this optical setting, $\mathcal{PT}$ symmetry was observed for the first time [5–7]. In addition, it has been introduced into many other physical disciplines such as Bose–Einstein condensates, electronic circuits and mechanical systems [8–13]. $\mathcal{PT}$ systems feature a unique property — phase transition, where the linear spectrum changes from all-real to partially-complex when the system parameters cross a certain threshold [3,4,14,15]. This phase transition has led to interesting applications such as single-mode $\mathcal{PT}$ lasers and unidirectional reflectionless optical devices [16–18]. A surprising property of $\mathcal{PT}$ systems is that, even though they are dissipative due to the gain and loss, they exhibit many properties of conservative systems, such as all-real linear spectra and continuous families of stationary nonlinear modes [3,4,9,14,15,19–26]. Thus, $\mathcal{PT}$ systems break the boundary between conservative and dissipative systems and offer novel wave-guiding possibilities. In addition, $\mathcal{PT}$ systems make loss useful, which is enlightening.

The downside of $\mathcal{PT}$ symmetry stems from the restrictive conditions set on the gain-loss profile, which must be odd. To overcome this restriction, non-$\mathcal{PT}$-symmetric dissipative systems sharing the properties of $\mathcal{PT}$-symmetric systems have been pursued. For instance, wide classes of non-$\mathcal{PT}$-symmetric potentials with all-real spectra were reported in [27–29,31,32]. In addition, it was discovered that in a certain class of such potentials with the form $g^2(x) + ig'(x)$, where $g(x)$ is an arbitrary real function, solitons also appear as continuous families, which is very counter-intuitive [31,33–35]. Furthermore, it was argued in [35] that potentials of the form $g^2(x) + ig'(x)$ are the only one-dimensional (1D) non-$\mathcal{PT}$-symmetric potentials which support soliton families. However, stability properties of these soliton families are still largely unknown, except for some evolution simulations of perturbed simple-shaped solitons in a certain non-$\mathcal{PT}$-symmetric potential below a phase transition in [31], which suggest that those simple solitons could be stable.

In this paper, we systematically study the linear stability of various soliton families in 1D nonlinear Schrödinger (NLS) equations with non-$\mathcal{PT}$-symmetric complex potentials both below and above phase transition. This study is performed by numerically computing the linear-stability spectra of these solitons. We

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show that both simple-shaped and multi-humped soliton families can be linearly stable in a wide range of parameter values below and above a phase transition. In addition, a pseudo-Hamiltonian–Hopf bifurcation is revealed, where pairs of purely-imaginary eigenvalues in the linear-stability spectra of solitons collide and bifurcate off the imaginary axis, creating oscillatory instability, which resembles Hamiltonian–Hopf bifurcations of solitons in Hamiltonian systems even though the present system is non-Hamiltonian. Our most important numerical finding is that, eigenvalues of the linear-stability operator of these solitons appear in quartets $(\lambda, -\lambda, \lambda^*, -\lambda^*)$, similar to conservative systems and $\mathcal{PT}$-symmetric systems. This quartet eigenvalue symmetry is very surprising for non-$\mathcal{PT}$-symmetric dissipative systems, and its consequences on the linear stability of these solitons are explained.

2. Preliminaries

The mathematical model we consider is the following NLS equation with an external potential

$$i\Psi_t + \Psi_{xx} + V(x)\Psi + \sigma|\Psi|^2\Psi = 0, \tag{2.1}$$

where $V(x)$ is a complex potential, and $\sigma = \pm 1$ is the sign of nonlinearity. This model governs nonlinear light propagation in an optical medium with gain and loss [4,36,37], as well as dynamics of Bose–Einstein condensates in a double-well potential with atoms injected into one well and removed from the other well [11,12,38]. If the potential $V(x)$ is real, Eq. (2.1) is conservative and Hamiltonian, and its properties have been investigated in numerous articles for many decades [36,37]. If $V(x)$ is complex but $\mathcal{PT}$-symmetric, i.e., $V^*(x) = V(-x)$, where the superscript * represents complex conjugation, then this $\mathcal{PT}$-symmetric system has been heavily studied in the last eight years [1,2]. If $V(x)$ is complex and non-$\mathcal{PT}$-symmetric, this equation is currently at the frontier of research. For non-$\mathcal{PT}$-symmetric potentials of the form

$$V(x) = g^2(x) + 2\gamma g(x) + ig'(x), \tag{2.2}$$

where $g(x)$ is a real asymmetric function and γ a real constant, the linear spectrum of the potential can be all-real, which is unusual [30–32]. Note that this form of the potential is equivalent to $g^2(x) + ig'(x)$ under a shift $g(x) + \gamma \to g(x)$ and a gauge transformation to Eq. (2.1). It is used in this article since it is more convenient to induce a phase transition by varying the parameter γ while keeping the function $g(x)$ fixed. A more important phenomenon with the potential (2.2) is that, Eq. (2.1) under this potential admits continuous families of solitons [31,34,35]. This is surprising since, in typical dissipative systems, solitons exist as isolated solutions with discrete power levels due to the requirement of balance between gain and loss [39]. Dissipative but $\mathcal{PT}$-symmetric systems admit soliton families with continuous power levels, which is interesting [9,14,19–24]. However, the existence of such soliton families can be easily understood due to the $\mathcal{PT}$ symmetry, which assures the balancing of gain and loss for all $\mathcal{PT}$-symmetric solitons [40]. Soliton families in non-$\mathcal{PT}$-symmetric systems, on the other hand, are much less obvious, and their existence has yet to be fully understood.

Solitons in Eq. (2.1) are of the form

$$\Psi(x,t) = e^{-i\mu t}\psi(x), \tag{2.3}$$

where μ is a real propagation constant, and $\psi(x)$ is a localized function satisfying the equation

$$\psi_{xx} + \mu\psi + V(x)\psi + \sigma|\psi|^2\psi = 0. \tag{2.4}$$

For the complex non-$\mathcal{PT}$-symmetric potential (2.2), these solitons exist as continuous families, and they can be computed by various numerical methods such as the squared-operator method and the Newton-conjugate-gradient method [36]. To study their linear stability, we perturb these solitons by infinitesimal normal modes,

$$\Psi(x,t) = e^{-i\mu t}\left[\psi(x) + f_1(x)e^{\lambda t} + f_2^*(x)e^{\lambda^* t}\right], \tag{2.5}$$

where $|f_1|, |f_2| \ll |\psi|$. Substituting this perturbation into Eq. (2.1) and linearizing, we obtain a linear-stability eigenvalue problem

$$L\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \lambda \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{2.6}$$

where

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{12}^* & L_{11}^* \end{pmatrix}, \tag{2.7}$$

and

$$L_{11} = i\left[\partial_{xx} + \mu + V(x) + 2\sigma|\psi|^2\right], \quad L_{12} = i\sigma\psi^2.$$

This eigenvalue problem can be computed by the Fourier collocation method (for the full spectrum) or the Newton-conjugate-gradient method (for individual discrete eigenvalues) [36]. If eigenvalues with positive real parts exist, the soliton is linearly (spectrally) unstable; otherwise it is linearly (spectrally) stable.

Symmetry properties of the linear-stability operator L and its eigenvalues are important since they strongly influence the stability results. If the potential $V(x)$ is real [i.e., when Eq. (2.1) is Hamiltonian], then L satisfies the following two symmetry relations,

$$L^* = \sigma_1 L \sigma_1^{-1}, \tag{2.8}$$

$$L^\dagger = -\sigma_3 L \sigma_3^{-1}, \tag{2.9}$$

where the superscript † represents the Hermitian (conjugate transpose) of a matrix operator, and

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are the first and third Pauli spin matrices. The similarity relation (2.8) shows that L^* and L share the same spectrum. Then, since the spectrum of L^* is also the complex conjugate of the spectrum of L, we see that eigenvalues of L must come in conjugate pairs (λ, λ^*). The symmetry relation (2.9) shows that the spectrum of $L^\dagger$ is negative of the spectrum of L. Since the spectrum of $L^\dagger$ is also complex conjugate of the spectrum of L, eigenvalues of L then must come in pairs of $(\lambda, -\lambda^*)$. Combining these two eigenvalue symmetries, we conclude that for real potentials (Hamiltonian systems), complex eigenvalues of L must come in quartets $(\lambda, -\lambda, \lambda^*, -\lambda^*)$, which is a well-known fact. In the special case when the eigenvalue λ is real or purely-imaginary, this quartet symmetry reduces to a pair symmetry $(\lambda, -\lambda)$.

It is noted that in a real potential $V(x)$, if the soliton $\psi(x)$ is also real (which is generally the case), then $L^* = -L$. Using this symmetry, instead of (2.9), one can also show that eigenvalues of L come in pairs of $(\lambda, -\lambda^*)$. However, for real potentials V in higher spatial dimensions, if the soliton ψ is complex (such as a vortex soliton), then the symmetry $L^* = -L$ would not hold, but (2.9) still does.

If the potential $V(x)$ is complex but $\mathcal{PT}$-symmetric, then the symmetry relation (2.8) persists, but the other relation (2.9) breaks down. In this case, if the soliton $\psi(x)$ is also $\mathcal{PT}$-symmetric, i.e., $\psi^*(x) = \psi(-x)$, then another symmetry relation

$$L^* = -\mathcal{P}L\mathcal{P}^{-1} \tag{2.10}$$

is valid, where $\mathcal{P}$ is the parity operator, i.e., $\mathcal{P}f(x) \equiv f(-x)$. Utilizing the two symmetry relations (2.8) and (2.10) and repeating

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