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Geometry of fast magnetosonic rays, wavefronts and shock waves

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ABSTRACT

Fast magnetosonic waves in a two-dimensional plasma are studied in the geometrical optics approximation. The geometry of rays and wavefronts influences decisively the formation and ulterior evolution of shock waves. It is shown that the curvature of the curve where rays start and the angle between rays and wavefronts are the main parameters governing a wide variety of possible outcomes.

1. Introduction

While the model of ideal magnetohydrodynamics represents the simplest description of the evolution of a neutral plasma, and both its weaknesses and its strengths are well known, the long term behavior of solutions is anything but easy to predict. In common with all nonlinear hyperbolic systems, shocks may develop and indeed do in many physically relevant situations, but their location and later evolution are extremely complex problems. However, for high frequency perturbations the methods of nonlinear geometric optics provide a more amenable analytical approach. While its philosophy is highly classical [1,2], rigorous mathematical justifications are more recent and in fact continue to this day [3–5]. There exists a vast bibliography for this technique and its applications [6–8], e.g. in elasticity [9], fluid dynamics [10,11] and ideal MHD [12,13]. The most desirable case occurs when dealing with waves of a single phase. When one admits superposition of waves whose phases are different solutions of the eikonal equation, resonance may occur [14,15] and the waves interact in unsuspected ways. Methods to deal with particular cases have been successfully applied e.g. to the two-dimensional Euler equation [5]. We will assume a single phase and make use of two excellent survey articles [16,17]. Even in this case most specific results assume dependence on a single spatial variable (although the system itself may be multidimensional). This way rays are straight and parallel lines and there is no trouble with their intersection. We wish to analyze a genuinely multidimensional case, keeping the remaining data as simple as possible; thus we consider propagation of fast magnetohydrodynamic waves in a static plane equilibrium: density, pressure and magnetic field are constant. Rays are straight lines and their angle with wavefronts is constant along each ray. Nevertheless, setting the location of the initial perturbation along an arbitrary curve in the plane, we allow for rays to converge generating caustics, and the wavefront normal also differ among different rays. The crucial parameters are precisely the curvature of the original curve, and also the variation along it of the angle between the static magnetic field and the normal. A very lengthy calculation shows that the first order term for the asymptotic expansion of the solution satisfies a differential equation along the rays which may be reduced to the Burgers equation, whose behavior is well understood. In particular the time of shock formation and the ulterior evolution of the shock wave are widely available in the literature e.g. in [7,8] and specially in [18]. However, the necessary changes both of variables and functions to reduce our problem to a Burgers form depend on the sign and relative size of the equilibrium quantities, plus the data in the original curve, so the admirable universality of the Burgers solution (which tends always to an N-wave) gives rise to a surprising variety of possible outcomes for the velocity, the shock strength and the overall shape of this wave. A final word of caution related to the intrinsic limitations of nonlinear geometrical optics. The evolution of the shock along each ray is governed by the Rankine–Hugoniot relations, but there is no guarantee that the final solution, transported through different rays, will satisfy also the Rankine–Hugoniot relations in the transverse direction to these rays. To achieve this further constraints in the original values would be necessary. Although the wavelength of our solutions is small, when rays approach one another interference occurs, which is not covered by geometrical optics; obviously for diverging rays the approximation is excellent.

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2. Geometry of rays

Consider a quasilinear hyperbolic system, written in the Einstein notation

$$\frac{\partial \mathbf{u}}{\partial t} + A_j(\mathbf{x},\mathbf{u})\frac{\partial \mathbf{u}}{\partial x_j} + \mathbf{C}(\mathbf{x},\mathbf{u}) = \mathbf{0}. \tag{1}$$

For any spatial vector $\mathbf{k}$ and equilibrium state $\mathbf{u}_0$ take a fixed eigenvalue $\Lambda(\mathbf{k})$,

$$\det(\Lambda(\mathbf{k})I + A_j(\mathbf{x},\mathbf{u}_0)k_j) = 0. \tag{2}$$

The eikonal equation associated to this eigenvalue is

$$\frac{\partial \phi}{\partial t} = \Lambda(\nabla\phi), \tag{3}$$

and ϕ is the phase. In our case the system will be the ideal MHD one, and we choose for Λ the fast magnetosonic frequency (see e.g. [19]). If $\mathbf{u}_0$ corresponds to a static state with pressure P, density ρ and magnetic field $\mathbf{B}$,

$$\Lambda(\mathbf{k})^2 = \frac{1}{2}\left(\frac{\partial P}{\partial \rho} + \frac{B^2}{\rho}\right)|\mathbf{k}|^2 + \frac{1}{2}\left[\left(\frac{\partial P}{\partial \rho} + \frac{B^2}{\rho}\right)^2 |\mathbf{k}|^4 - 4\frac{\partial P}{\partial \rho}\frac{(\mathbf{B}\cdot\mathbf{k})^2}{\rho}|\mathbf{k}|^2\right]^{1/2}. \tag{4}$$

Rays are solutions of the system

$$\frac{d\mathbf{x}}{dt} = \nabla_{\mathbf{k}}\Lambda(\mathbf{x},\mathbf{k})$$
$$\frac{d\mathbf{k}}{dt} = -\nabla_{\mathbf{x}}\Lambda(\mathbf{x},\mathbf{k}). \tag{5}$$

The phase is constant along rays,

$$\frac{d}{dt}(\phi(t,\mathbf{x}(t))) = 0. \tag{6}$$

Often one takes a normalized vector $\mathbf{n} = \mathbf{k}/|\mathbf{k}|$ and uses the frequency

$$c(\mathbf{n}) = \frac{\Lambda(\mathbf{k})}{|\mathbf{k}|}. \tag{7}$$

Equations (5) for the plane may be written in terms of c, $\mathbf{n}$ and its orthogonal $\mathbf{n}^\perp$, chosen so that $\{\mathbf{n},\mathbf{n}^\perp\}$ form an orthonormal positive system:

$$\frac{d\mathbf{x}}{dt} = c\mathbf{n} + (\mathbf{n}^\perp\cdot\nabla_{\mathbf{n}}c)\mathbf{n}^\perp, \tag{8}$$
$$\frac{d\mathbf{n}}{dt} = -(\mathbf{n}^\perp\cdot\nabla_{\mathbf{x}}c)\mathbf{n}^\perp. \tag{9}$$

For static equilibria, the fast magnetosonic frequency $c(\mathbf{n})$ satisfies

$$c(\mathbf{n})^2 = \frac{1}{2}\left(\frac{\partial P}{\partial \rho} + \frac{B^2}{\rho}\right) + \frac{1}{2}\left[\left(\frac{\partial P}{\partial \rho} + \frac{B^2}{\rho}\right)^2 - 4\frac{\partial P}{\partial \rho}\frac{(\mathbf{B}\cdot\mathbf{n})^2}{\rho}\right]^{1/2}. \tag{10}$$

This equation may be written in terms of the speed of sound $c_s^2 = \partial P/\partial\rho$, Alfvén speed $c_A^2 = B^2/\rho$, and the angle θ that forms the magnetic field $\mathbf{B}$ with $\mathbf{n}$:

$$c(\mathbf{n})^2 = \frac{1}{2}(c_s^2 + c_A^2) + \frac{1}{2}\left[(c_s^2 + c_A^2)^2 - 4c_s^2 c_A^2\cos^2\theta\right]^{1/2}. \tag{11}$$

From now on we will consider a static ideal MHD equilibrium where both magnetic field and density are constant in space. In this case $c(\mathbf{n})$ does not depend on $\mathbf{x}$, so that we find from (9) that $\mathbf{n}$ (and $\mathbf{n}^\perp$) are constant along the ray; and since both coefficients in (8) are constant, rays are straight lines. Denoting by $\mathbf{b}$ the unit magnetic field, by our definition of θ

$$\mathbf{n} = \cos\theta\,\mathbf{b} + \sin\theta\,\mathbf{b}^\perp$$
$$\mathbf{n}^\perp = -\sin\theta\,\mathbf{b} + \cos\theta\,\mathbf{b}^\perp, \tag{12}$$

which implies

$$\frac{d\mathbf{n}}{d\theta} = (-\sin\theta)\,\mathbf{b} + (\cos\theta)\,\mathbf{b}^\perp = \mathbf{n}^\perp, \tag{13}$$

so that, writing as in (11) c as a function of θ (all the rest being constants), and denoting by c' the derivative of c with respect to θ,

$$\frac{dc}{d\theta} = c'(\theta) = \frac{d\mathbf{n}}{d\theta}\cdot\nabla_{\mathbf{n}}c = \mathbf{n}^\perp\cdot\nabla_{\mathbf{n}}c, \tag{14}$$

so that (8) may be written

$$\frac{d\mathbf{x}}{dt} = c\mathbf{n} + c'\mathbf{n}^\perp. \tag{15}$$

We see from (5) that $\mathbf{k} = \mathbf{k}_0$ is constant along the ray, and since $\nabla\phi = \mathbf{k}_0$, this is also constant along the ray, as well as

$$\frac{\partial\phi}{\partial t} = -c|\mathbf{k}_0|. \tag{16}$$

Let us fix a single ray, and call $\boldsymbol{\alpha} = c\mathbf{n} + c'\mathbf{n}^\perp$. Then the ray is given by

$$\mathbf{x}(t) = \boldsymbol{\alpha}t + \mathbf{x}(0). \tag{17}$$

Choosing as parameter the arc length s instead of t so that we may reserve this for the time,

$$\mathbf{x}(s) = \frac{\boldsymbol{\alpha}}{|\boldsymbol{\alpha}|}s + \mathbf{x}(0). \tag{18}$$

Since

$$\frac{d}{dt}\phi(t,s(t)) = \frac{\partial\phi}{\partial t} + |\boldsymbol{\alpha}|\frac{\partial\phi}{\partial s} = 0, \tag{19}$$

we find the simple expression for the phase in a single ray

$$\phi(t,s) = c\frac{|\mathbf{k}_0|}{|\boldsymbol{\alpha}|}(s - |\boldsymbol{\alpha}|t) + \text{const}. \tag{20}$$

Abbreviating $|\boldsymbol{\alpha}| = \alpha$, and labeling $\phi(0,0) = 0$, we may write

$$\phi(t,s) = A(s - \alpha t). \tag{21}$$

To set the ray geometry appropriate to our problem, we start from a curve $\mathbf{g}$ parametrized by the arc length ξ, $\xi \in (\xi_0 - \epsilon, \xi_0 + \epsilon)$, and consider rays orthogonal to this curve. Let us therefore take a normal unitary vector $\mathbf{T}$, chosen so that $\mathbf{g}'(\xi) = \mathbf{T}^\perp(\xi)$, and $\mathbf{T}$, $\mathbf{T}^\perp$ form a positive orthonormal system. Thus the ray starting at $\mathbf{g}(\xi)$ may be parametrized by the arc length s as $s \to \mathbf{g}(\xi) + s\mathbf{T}(\xi)$. It is easy to see that the transported curves $\mathbf{g}_s : \xi \to \mathbf{g}(\xi) + s\mathbf{T}(\xi)$ form with the rays a family of orthogonal curves for as long as there is no self-intersection. While (ξ, s) form a global family of orthogonal coordinates for the area covered by these curves, the fact that ξ is not the arc length in the curve $\mathbf{g}_s$ makes us to choose r, the arc length on this curve, starting at $(r = 0)$ (i.e. $\xi = \xi_0$), as new variable. We see that $r = r(\xi, s)$. Since $\mathbf{T}^\perp$ is the tangent vector to $\mathbf{g}_s$ and $\mathbf{T}$ is minus the normal vector,

$$\frac{d\mathbf{T}^\perp}{dr} = -\kappa\mathbf{T} \qquad \frac{d\mathbf{T}}{dr} = \kappa\mathbf{T}^\perp, \tag{22}$$

where $\kappa = \kappa(r,s)$ represents the curvature of $\mathbf{g}_s$. Let us find $\kappa(r,s)$ in terms of the curvature of the original curve $\kappa(r,0)$. We have

$$\mathbf{g}_s' : r \to \mathbf{g}'(r) + s\mathbf{T}'(r) = \mathbf{T}^\perp(r) + s\kappa(r,0)\mathbf{T}^\perp(r)$$
$$\mathbf{g}_s'' : r \to -\kappa(r,0)\mathbf{T}(r) + s\frac{\partial\kappa}{\partial r}(r,0)\mathbf{T}^\perp(r) - s\kappa(r,0)^2\mathbf{T}(r). \tag{23}$$

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