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Pseudo-Newtonian planar circular restricted 3-body problem

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ABSTRACT

We study the dynamics of the planar circular restricted three-body problem in the context of a pseudo-Newtonian approximation. By using the Fodor–Hoenselaers–Perjés procedure, we perform an expansion in the mass potential of a static massive spherical source up to the first non-Newtonian term, giving place to a gravitational potential that includes first-order general relativistic effects. With this result, we model a system composed by two pseudo-Newtonian primaries describing circular orbits around their common center of mass, and a test particle orbiting the system in the equatorial plane. The dynamics of the new system of equations is studied in terms of the Poincaré section method and the Lyapunov exponents, where the introduction of a new parameter ϵ, allows us to observe the transition from the Newtonian to the pseudo-Newtonian regime. We show that when the Jacobian constant is fixed, a chaotic orbit in the Newtonian regime can be either chaotic or regular in the pseudo-Newtonian approach. As a general result, we find that most of the pseudo-Newtonian configurations are less stable than their Newtonian equivalent.

1. Introduction

One of the simplest and most frequently studied version of the general three-body problem, is the planar circular restricted three-body problem (henceforth CRTBP), which can be stated as follows:

- Two primaries, $\mathcal{M}_1$ and $\mathcal{M}_2$ at positions X_1 and X_2, respectively, follow a circular orbit around their common center of mass keeping a fixed distance r, while moving at constant angular velocity ω_0.
- A third body $\mathcal{M}$, that is much smaller than either $\mathcal{M}_1$ or $\mathcal{M}_2$, remains in the orbital plane of the primaries.
- The equations of motion are derived only for the test particle $\mathcal{M}$, whose motion does not affect the primaries.

The basic formulation of the CRTBP dates back to Euler, who proposed the use of synodical coordinates (x, y) instead of the inertial coordinate system (X, Y), in order to simplify this problem [1]. The transformation between these two systems can be performed by means of the rotation matrix,[1]

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\omega_0 t & -\sin\omega_0 t \\ \sin\omega_0 t & \cos\omega_0 t \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}. \quad (1)$$

Using this transformation, Lagrange proved the existence of five equilibrium points for the system, named Lagrangian points. The subject of equilibrium points in the CRTBP has been studied extensively in the literature (see *e.g.* [2] and references therein). The discovery of the Trojan asteroids around the Lagrangian points L_4 and L_5 in the Sun–Jupiter system [3], and the recent observations of asteroids around L_4 for the Sun–Earth system [4], increased theoretical research on the subject (see *e.g.* [5]). It should be noted that, in spite of the fact that the CRTBP is much simpler than the general three-body problem, it is non-integrable, which opened the possibility to analyze systematically the orbits [2,6].

Under the assumption of weak fields and low velocities, and as a first approximation to the relativistic CRTBP, in 1967 [7] Krefetz considered for the first time the post-Newtonian equations of motion for the CRTBP, using the Einstein–Infeld–Hoffmann (EIH) formalism [8]. The Lagrangian for this system was explicitly presented by Contopoulos in 1976 [9] and some typos for the Jacobian constant were corrected by Maindl et al. [10], who also studied the deviations due to the post-Newtonian corrections on the Lagrangian points [11]. Concerning analytical solutions to the general relativistic three-body problem, Yamada et al. [12] obtained a collinear solution by using the EIH approximation up to the first order. In a later paper, they studied the post-Newtonian triangular solution for three finite masses, showing that such configuration is

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[1] Along the paper $G = \mathcal{M} = \omega_0 = r = 1$, is understood.

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not always equilateral [13]. Recently, as a first study of chaos in the post-Newtonian CRTBP, Huang and Wu [14] studied the influence of the separation between the primaries, concluding that if it is close enough, the post-Newtonian dynamics is qualitatively different. In particular, some Newtonian bounded orbits become unstable.

To avoid the cumbersome equations of motion that take place in the post-Newtonian formalism, Steklain and Letelier used the Paczyński–Wiita pseudo-Newtonian potential to study the dynamics of the CRTBP in the Hill's approximation [15], finding that some pseudo-Newtonian systems are more stable than their Newtonian counterparts. Following this idea, and considering that the Jacobian constant is not preserved in the post-Newtonian approximation (which limits the dynamical studies), in the present paper we shall use an alternative approach to studying the dynamics of the pseudo-Newtonian CRTBP. To do so, we derive an approximate potential for the gravitational field of two uncharged spinless particles modeled as sources with multipole moment m, by using the Fodor–Hoenselaers–Perjés (FHP) procedure [16] (taking into account the corrections made by Sotiriou and Apostolatos [17]). Abusing astrophysical terminology, we call the new potential pseudo-Newtonian, due to the fact that in this kind of approaches the common Newtonian formulas are used even when the resulting potentials do not satisfy the Laplace equation. Unlike other pseudo-Newtonian approaches, the final expressions are not adhoc proposals but are derived directly from the multipole structure of the source.

The paper is organized as follows: In section 2, by means of the FHP procedure, we calculate the gravitational pseudo-potential for each primary; then we write down the Lagrangian of the CRTBP with their respective equations of motion for a test particle under the influence of this potential. In section 3, we analyze the gradual transition of the dynamics for the FHP pseudo-Newtonian approximation to the classical regime. The analysis is made using Poincaré surfaces of section and the variational method for the calculation of the largest Lyapunov exponent [18]. Finally, in section 4 we summarize our main conclusions.

2. Pseudo-Newtonian equations of motion

The Fodor–Hoenselaers–Perjés procedure is an algorithm to determine the multipole moments of stationary axisymmetric electrovacuum space–times [16]. The method can be stated as follows: In the Ernst formalism [19,20], Einstein field equations are reduced to a pair of complex equations through the introduction of the complex potentials $\mathcal{E}$ and Ψ, which can be defined in terms of the new potentials ξ and ς, through the definitions

$$\mathcal{E} = \frac{1-\xi}{1+\xi}, \quad \Psi = \frac{\varsigma}{1+\xi}, \tag{2}$$

satisfying the alternative representation of the Einstein–Maxwell field equations,

$$(\xi\xi^* - \varsigma\varsigma^* - 1)\nabla^2\xi = 2(\xi^*\nabla\xi - \varsigma^*\nabla\varsigma)\cdot\nabla\xi, \tag{3}$$

$$(\xi\xi^* - \varsigma\varsigma^* - 1)\nabla^2\varsigma = 2(\xi^*\nabla\xi - \varsigma^*\nabla\varsigma)\cdot\nabla\varsigma. \tag{4}$$

The fields ξ and ς are related to the gravitational and electromagnetic potentials in a very direct way,

$$\xi = \phi_M + i\phi_J, \quad \varsigma = \phi_E + i\phi_H, \tag{5}$$

where ϕ_α with $\alpha = M, J, E, H$, are analogous to the Newtonian mass, angular momentum, electrostatic and magnetic potentials, respectively (see *e.g.* [21] and [17]). Hereafter, for the sake of convenience, we consider $\phi_E = \phi_H = 0$, which implies $\varsigma = \Psi = 0$, *i.e.* the absence of electromagnetic fields.

Now, according to Geroch and Hansen [21,22], the multipole moments of a given space–time are defined by measuring the deviation from flatness at infinity. Following this idea, the initial 3-metric h_{ij} is mapped to a conformal one, that is $h_{ij} \to \tilde{h}_{ij} = \Omega^2 h_{ij}$. The conformal factor Ω transforms the potential ξ into $\tilde{\xi} = \Omega^{-1/2}\xi$, with $\Omega = \bar{r}^2 = \bar{\rho}^2 + \bar{z}^2$, and

$$\bar{\rho} = \frac{\rho}{\rho^2+z^2}, \quad \bar{z} = \frac{z}{\rho^2+z^2}, \quad \bar{\varphi} = \varphi. \tag{6}$$

On the other hand, the potential $\tilde{\xi}$ can be written in a power series of $\bar{\rho}$ and $\bar{z}$ as

$$\tilde{\xi} = \sum_{i,j=0}^{\infty} a_{ij}\bar{\rho}^i\bar{z}^j, \tag{7}$$

and the coefficients a_{ij} are calculated by the recursive relations [17]

$$\begin{aligned}(r+2)^2 a_{r+2,s} = &-(s+2)(s+1)a_{r,s+2} \\ &+ \sum_{k,l,m,n,p,g}(a_{kl}a^*_{mn} - b_{kl}b^*_{mn})[a_{pg} \\ &\times (p^2+g^2-4p-5g-2pk-2gl-2) \\ &+ a_{p+2,g-2}(p+2) \\ &\times (p+2-2k) \\ &+ a_{p-2,g+2}(g+2)(g+1-2l)],\end{aligned} \tag{8}$$

where $m = r-k-p$, $0 \le k \le r$, $0 \le p \le r-k$, with k and p even, and $n = s-l-g$, $0 \le l \le s+1$, and $-1 \le g \le s-l$. Finally, the gravitational multipole moments P_i of the source are computed from their values on the symmetry axis $m_i \equiv a_{0i}$, by means of the following relationships

$$\begin{aligned}&P_0 = m_0, \quad P_1 = m_1, \quad P_2 = m_2, \\ &P_3 = m_3, \quad P_4 = m_4 - \frac{1}{7}m_0^*M_{20}, \\ &P_5 = m_5 - \frac{1}{3}m_0^*M_{30} - \frac{1}{21}m_1^*M_{20}\end{aligned} \tag{9}$$

where $M_{ij} = m_i m_j - m_{i-1}m_{j+1}$.

From the previous, it can be inferred that once we know the gravitational multipole moments, and following the inverse procedure, it is possible to determinate approximate expressions for the gravitational potential ξ, in terms of the physical parameters of the source (see *e.g.* [23]). Hence, let us apply the outlined procedure to a concrete example, a source whose multipole structure is given by

$$P_0 = m, P_i = 0 \quad \text{for} \quad i \ge 1, \tag{10}$$

such that m denotes the mass of the source. From Eq. (8) and the seed $a_{00} = m$, it can be noted that the only non-vanishing coefficients are $a_{2n,2m}$ with $n, m \in \mathbb{N}$. Thus the potential ξ is reconstructed from (7), (6) and $\xi = \Omega^{1/2}\tilde{\xi}$, and is explicitly given by

$$\begin{aligned}\xi(\rho,z) = &\frac{m}{\sqrt{\rho^2+z^2}} - \frac{m^3\rho^2}{2\left(\rho^2+z^2\right)^{5/2}} \\ &+ \frac{m^5\rho^2\left(3\rho^2-4z^2\right)}{8\left(\rho^2+z^2\right)^{9/2}} + \ldots\end{aligned} \tag{11}$$

It is important to note that the infinite sum of terms of the potential ξ corresponds to the Schwarzschild solution, while the lower order of approximation, *i.e.* by taking the first term, corresponds to the Newtonian potential. In order to stress that the approximations

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