Physics Letters A 381 (2017) 597–603

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Hindmarsh–Rose model: Close and far to the singular limit

Roberto Barrio [a,b,*], Santiago Ibáñez [c], Lucía Pérez [c]

[a] Departamento de Matemática Aplicada and IUMA, University of Zaragoza, E-50009 Zaragoza, Spain
[b] Computational Dynamics group, University of Zaragoza, E-50009 Zaragoza, Spain
[c] Departamento de Matemáticas, University of Oviedo, E-33007 Oviedo, Spain

A R T I C L E   I N F O

Article history:
Received 12 August 2016
Received in revised form 9 December 2016
Accepted 10 December 2016
Available online 15 December 2016
Communicated by C.R. Doering

Keywords:
Hindmarsh–Rose model
Singular limit
Slow-fast dynamics
Spike-adding

A B S T R A C T

Dynamics arising in the Hindmarsh–Rose model are considered from a novel perspective. We study qualitative changes that occur as the time scale of the slow variable increases taking the system far from the slow-fast scenario. We see how the structure of spike-adding still persists far from the singular case but the geometry of the bifurcations changes notably. Particular attention is paid to changes in the shape of the homoclinic bifurcation curves and the disappearance of Inclination-Flip codimension-two points. These transformations seem to be linked to the way in which the spike-adding takes place, the changing from fold/hom to fold/Hopf bursting behavior and also with the way in which the chaotic regions evolve as the time scale of the slow variable increases.

1. Introduction

It is out of discussion that to understand such a complex mechanism as the brain, and in general any living neural network, it is compulsory to know first the working of its basic building blocks: the neurons. Since the seminal contribution of Hodgkin and Huxley [1], neurons are commonly viewed as dynamical systems. Elements of bifurcation theory play an essential role in this context and help to understand neuronal activity.

The range of activity types that a neuron can exhibit is quite broad and includes quiescence (the state of not firing), tonic spiking, bursting and irregular (or chaotic) spiking. Each of these behaviors has its counterpart in the language of dynamical systems, either as stable periodic or chaotic orbits. Even the process of spike-adding can be linked to specific codimension-two homoclinic bifurcations (Orbit-Flip and Inclination-Flip points) and also to the so called canard explosions [2,3].

Hindmarsh–Rose (HR in the sequel) equations

$$\begin{cases} x' = y - ax^3 + bx^2 + I - z, \\ y' = c - dx^2 - y, \\ z' = \varepsilon(s(x - x_0) - z) \end{cases} \tag{1}$$

were introduced in [4] as a reduction of the Hodgkin–Huxley model. The HR model is simpler but it captures the main dynamical behaviors which are displayed by real neurons: quiescence, tonic spiking, bursting and irregular spiking (see [5–13]). The system possesses two time scales: x and y evolve as fast variables while z does it as a slow variable (so, it is a slow-fast dynamical system). The x variable should be treated as the voltage across the cell membrane, while the y and z variables would describe kinetics of some ionic currents. The *small* parameter ε controls the time scale of z and x_0 controls the rest potential of the system.

Different choices of the parameters have been considered in the literature (see [11] for an excellent review of the dynamics of the model). Following [2,11,12] we assume that

$$a = 1, \quad c = 1, \quad d = 5, \quad s = 4, \quad \text{and} \quad x_0 = -1.6. \tag{2}$$

With this choice, (1) becomes a family dependent only on parameters (b, I, ε). These parameters will be our primary bifurcation parameters.

In this paper we pay attention to the changes in the global picture as ε varies. From a realistic point of view it is clear that only small values of ε are of interest: typically $\varepsilon \ll 1$. We include a preliminary study about the singular limit of some relevant bifurcations. Nevertheless, in contrast with other approaches, we want to emphasize that the understanding of the bifurcation diagram for higher values of the slow time scale should be a crucial ingredient to get a whole picture of the dynamics and also it helps to understand what happens for $\varepsilon < 1$ (and not only $\varepsilon \ll 1$).

The article is arranged as follows: In Section 2 we compute the singular limit of the Hopf bifurcations as $\varepsilon \searrow 0$. Moreover, we show with numerical evidences that a singular limit also exists for the homoclinic bifurcation curves. We compare our results with

* Corresponding author at: Departamento de Matemática Aplicada and IUMA, University of Zaragoza, E-50009 Zaragoza, Spain.
E-mail addresses: rbarrio@unizar.es (R. Barrio), mesa@uniovi.es (S. Ibáñez), lpcuadrado@gmail.com (L. Pérez).

http://dx.doi.org/10.1016/j.physleta.2016.12.027
0375-9601/

those in [14–16] where similar singular limits were studied in a different model. In section 3 we investigate how the homoclinic bifurcation curves change as ε increases from small (slow-fast dynamics) to large values. We will see that, although the geometry of the bifurcation curves changes rapidly, many common features seem to persist. Section 4 is devoted to show how the global picture of spike-adding, bursting and chaotic behavior, bounded inside a loop formed by the Hopf bifurcation curves, evolves with ε. We will describe how this evolution seems to be linked to the changes along the homoclinic bifurcation curves. A summary is presented in Section 5. All continuations of bifurcation curves have been done with the free software AUTO [17,18].

2. Singular limits: Hopf bifurcation and homoclinic bifurcation

It easily follows (fixing all parameters but (b, I, ε)) that the equilibrium points of (1) are given by

$$y = 1 - 5x^2, \qquad z = 4(x + 1.6), \tag{3}$$

with x any real root of

$$P(x) = I - 5.4 - 4x + (b - 5)x^2 - x^3. \tag{4}$$

The Jacobian at a given equilibrium point is given by

$$\begin{pmatrix} -3x^2 + 2bx & 1 & -1 \\ -10x & -1 & 0 \\ 4\varepsilon & 0 & -\varepsilon \end{pmatrix}, \tag{5}$$

with characteristic polynomial

$$Q(\lambda) = \lambda^3 + q_2(x, b, \varepsilon)\lambda^2 + q_1(x, b, \varepsilon)\lambda + q_0(x, b, \varepsilon),$$

where

$$q_2(x, b, \varepsilon) = 3x^2 - 2bx + 1 + \varepsilon,$$
$$q_1(x, b, \varepsilon) = 3x^2 + (10 - 2b)x + \varepsilon(3x^2 - 2bx + 5),$$
$$q_0(x, b, \varepsilon) = \varepsilon\big(3x^2 + (10 - 2b)x + 4\big).$$

Necessary conditions for an Andronov–Hopf (AH) bifurcation are

$$P(x) = 0$$
$$C(x, b, \varepsilon) = q_2(x, b, \varepsilon)q_1(x, b, \varepsilon) - q_0(x, b, \varepsilon) = 0$$
$$q_1(x, b, \varepsilon) > 0.$$

The above conditions characterize a collection of surfaces on the space of parameters whose limit when $\varepsilon \searrow 0$ is given by

$$I - 5.4 - 4x + (b - 5)x^2 - x^3 = 0, \tag{6}$$
$$(3x^2 - 2bx + 1)\big(3x^2 + (10 - 2b)x\big) = 0, \tag{7}$$
$$3x^2 + (10 - 2b)x \geq 0. \tag{8}$$

Although the condition $q_1(x, b, \varepsilon) > 0$ is stated in terms of a strict inequality, we must consider the possibility of a non strict inequality at the limit when $\varepsilon \searrow 0$. The set S of points satisfying the above conditions consists of three curves as depicted (dashed blue) in Fig. 1. AH bifurcations curves for $\varepsilon = 0.005$ are also shown. Note that not the whole set S becomes the singular limit for AH bifurcations curves. When $x = 2(b - 5)/3$, (7) is satisfied and substituting in (6) we get the equation for the graph G of a polynomial $I(b)$ of degree 3. On the other hand, (8) is also satisfied because $3x^2 + (10 - 2b)x = 0$. It follows that G is the singular limit for a surface in the 3-parameter space satisfying $P(x) = 0$ and $C(x, b, \varepsilon) = 0$, but only a part of it satisfies $q_1(x, b, \varepsilon) > 0$. We note that the bifurcation diagram of the HR-model does not display a U-shaped Hopf bifurcation curve as that observed for other excitable systems (see [14–16,19]).

Fig. 1 also shows four homoclinic bifurcation curves (green and black) for different values of ε. The lowest value (black) is for $\varepsilon = 0.005$. In this case (for small values of ε), as for the systems considered in [14–16,19], homoclinic bifurcation curves are C-shaped. Numerical simulations show that there is a singular limit for the homoclinic bifurcations. Nevertheless, unlike the model studied in [15], a characterization of such singular limit involves extra difficulties and we pose this question as an open question for the next future. Anyway, it must be noticed that, as in [2,15], according to the numerical simulations, the homoclinic bifurcations curves do not terminate at a point approaching the set S. On the contrary, at both "ends" there is a sharp turning of the curve. However, this will make clear in the next section.

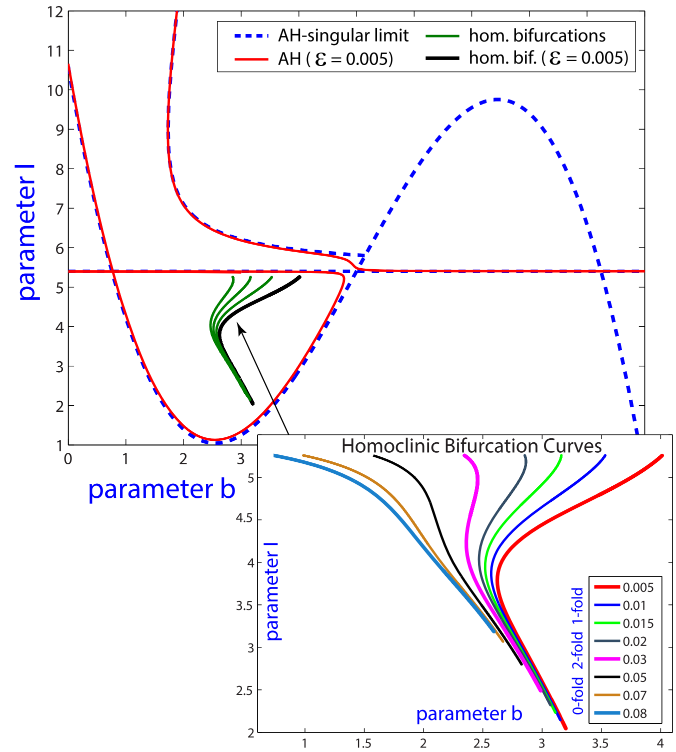

Fig. 1. Some features of the bifurcation diagram for the Hindmarsh–Rose model. Dashed blue curves show the set S of curves satisfying (6), (7) and (8), which contains the singular limit ($\varepsilon = 0$) of AH bifurcations in the full system. AH bifurcations (solid red) are shown for $\varepsilon = 0.005$. Note that not the whole set S is part of the singular limit. Homoclinic bifurcations (solid green and black) are shown for different values of ε (the lowest value (black) corresponds to $\varepsilon = 0.005$). In the magnification the first primary homoclinic bifurcations curves are shown for different values of ε. As the small parameter ε increases, the number of "visible" foldings (with respect to b) of the homoclinic curve changes. (For interpretation of the colors in this figure, the reader is referred to the web version of this article.)

3. Homoclinic bifurcations

Of course, as already argued, an essential piece to get the whole picture of the dynamics emerging in the HR-model is to understand the role of the singular limit as the source of a puzzling bifurcation diagram. Nevertheless, to have a deeper knowledge of the model, it is also crucial to study a wider range of time scales of the slow variable z. In this approach, the latest goal should be to find organizing centers located not necessarily close to the singular limit and to understand how the bifurcations evolve as ε decreases. Hence, from a different perspective, this approach could be helpful to give some insight into the global picture that we already know to be very entangled for $\varepsilon \ll 1$.

Since this paper focuses mainly on the role played by the homoclinic bifurcation, we study how they evolve as ε varies (in this paper we just show the first primary homoclinic orbits, related with the first spike-adding process [2]). The numerical results dis-