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An integrable semi-discretization of the Boussinesq equation

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ABSTRACT

In this paper, we present an integrable semi-discretization of the Boussinesq equation. Different from other discrete analogues, we discretize the 'time' variable and get an integrable differential-difference system. Under a standard limitation, the differential-difference system converges to the continuous Boussinesq equation such that the discrete system can be used to design numerical algorithms. Using Hirota's bilinear method, we find a Bäcklund transformation and a Lax pair of the differential-difference system. For the case of 'good' Boussinesq equation, we investigate the soliton solutions of its discrete analogue and design numerical algorithms. We find an effective way to reduce the phase shift caused by the discretization. The numerical results coincide with our analysis.

1. Introduction

The importance of soliton-producing nonlinear wave equations is well understood among theoretical physicists and applied mathematicians [1–3]. An equation that produces solitons is the well-known Boussinesq equation

$$u_{tt} - u_{xx} + \delta u_{xxxx} + 3\delta (u^2)_{xx} = 0. \tag{1.1}$$

With $\delta > 0$, equation (1.1) is referred to as the "good" Boussinesq (GB) equation [5] or the nonlinear beam equation [6]. And with $\delta < 0$ it is the "bad" Boussinesq (BB) equation which has been studied by Hirota [7,8].

We consider the equivalent form of the Boussinesq equation (1.1)

$$u_t = v_x, \tag{1.2}$$

$$v_t = u_x - \delta u_{xxx} - 3\delta (u^2)_x, \tag{1.3}$$

which we still call it Boussinesq equation without confusion. It possesses the following Lax pair,

$$L\Phi = \lambda\Phi, \tag{1.4}$$

$$\Phi_t = A\Phi, \tag{1.5}$$

where $L = 4\delta\partial_x^3 + (6\delta u - 1)\partial_x + (3\delta u_x + av)$, $A = a\partial_x^2 + au$ are differential operators and $a^2 = 3\delta$ is a constant. With

$$u = 2(lnf)_{xx}$$

the equation is transformed to

$$(D_t^2 - D_x^2 + \delta D_x^4) f \cdot f = 0, \tag{1.6}$$

where D is the bilinear operator defined by Hirota [4]. The Boussinesq equation has been studied for many years from various viewpoints including methods of finding soliton solutions [7,9–11], similarity reductions [12–15], linearization [16], Painlevé property [17, 18], numerical simulation [19–23] etc.

This paper is focused on the integrable discretization of the Boussinesq equation. The problem of integrable discretization is to discretize an integrable system, meanwhile preserving its integrability. Ablowitz and Ladik [24], Hirota [25–27], Nijhoff [28,29], Levi [31,32] and others did pioneer work on the problem of integrable discretization over 30 years ago. Other works include Suris' Hamiltonian approach [30], Schiff's loop group approach, etc. Here we have only given a few representative references. Other recent works can be found in [33–49] and references there.

In this paper, we start from the following bilinear equations,

$$(D_s^2 - D_x^2 + \delta D_x^4) f_m \cdot f_m = 0, \tag{1.7}$$

$$(aD_x D_s - \delta D_x^3) e^{\frac{1}{2}D_m} f_m \cdot f_m = \frac{2a}{k} D_x e^{\frac{1}{2}D_m} f_m \cdot f_m, \tag{1.8}$$

where $a^2 = 3\delta$ is a constant. With s replaced by t, equation (1.7) becomes the bilinear Boussinesq equation (1.6) and equation (1.8)

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is actually part of the Bäcklund transformation of Boussinesq equation [4,8]. It is a well-known idea that integrable discretizations are provided by a suitable interpretation of Bäcklund transformations [31,32]. Considering the compatibility between an integrable system and its Bäcklund transformation, it is quite reasonable to take the integrable equation and its Bäcklund transformation together as an extended system. Then the work left are to introduce a suitable discrete variable to make it a difference system and prove its integrability.

With transformation $u_m = 2(\ln f_m)_{xx}$, $v_m = 2(\ln f_m)_{xs}$, $w_m = 2(\ln f_m)_x$, $\eta_m = 2(\ln f_m)_s$, the bilinear equations are transformed to nonlinear equations (1.9)–(1.12). There is a standard limit between the differential-difference system and the continues Boussinesq equation (1.2)–(1.3). Actually, if we take $u_{m+1} = u(x, t + k)$, $v_{m+1} = v(x, t + k)$, $w_{m+1} = w(x, t + k)$, $\eta_{m+1} = \eta(x, t + k)$ and the limit $k \to 0$, equations (1.9)–(1.10) tend to the Boussinesq equation (1.2)–(1.3) and equations (1.11)–(1.12) become trivial equations:

$$\begin{aligned}\frac{2}{k}(u_{m+1}-u_m) &= [(v_{m+1,x}+v_{m,x})+\frac{1}{2}(u_{m+1}-u_m)(\eta_{m+1}-\eta_m)\\&\quad+\frac{1}{2}(w_{m+1}-w_m)(v_{m+1}-v_m)]\\&\quad-\frac{a}{3}[(u_{m+1,xx}-u_{m,xx})+\frac{3}{2}(u_{m+1}^2-u_m^2)\\&\quad+\frac{3}{2}(w_{m+1}-w_m)(u_{m+1,x}+u_{m,x})\\&\quad+\frac{3}{4}(w_{m+1}-w_m)^2(u_{m+1}-u_m)],\end{aligned}\tag{1.9}$$

$$\begin{aligned}\frac{2}{k}(v_{m+1}-v_m) &= [(u_{m+1,x}+u_{m,x})-\delta(u_{m+1,xxx}+u_{m,xxx})-3\delta(u_{m+1}^2+u_m^2)_x\\&\quad+\frac{1}{2}(v_{m+1}-v_m)(\eta_{m+1}-\eta_m)\\&\quad+\frac{1}{2}(u_{m+1}-u_m)(w_{m+1}-w_m)\\&\quad-\frac{\delta}{2}(w_{m+1}-w_m)(u_{m+1,xx}-u_{m,xx})\\&\quad-\frac{3\delta}{2}(w_{m+1}-w_m)(u_{m+1}^2-u_m^2)]\\&\quad-\frac{a}{3}[(v_{m+1,xx}-v_{m,xx})+\frac{3}{2}(v_{m+1}-v_m)(u_{m+1}+u_m)\\&\quad+\frac{3}{2}(w_{m+1}-w_m)(v_{m+1,x}+v_{m,x})\\&\quad+\frac{3}{4}(w_{m+1}-w_m)^2(v_{m+1}-v_m)],\end{aligned}\tag{1.10}$$

$$w_{m+1,x}-w_{m,x}=u_{m+1}-u_m,\tag{1.11}$$

$$\eta_{m+1,x}-\eta_{m,x}=v_{m+1}-v_m.\tag{1.12}$$

In [49], the authors have given a different integrable analogue

$$(D_t^2-D_z^2-D_z^4)f_n\cdot f_n=0,\tag{1.13}$$

$$(D_t-aD_z^2+\frac{2a}{h}D_z)e^{\frac{1}{2}D_n}f_n\cdot f_n=0.\tag{1.14}$$

This system is an integrable discretization of the spatial variable. With z replaced by x, the bilinear equation (1.13) gives to the bilinear Boussinesq equation (1.6). Meanwhile, equation (1.14), comparing with equation (1.8), becomes another part of the Bäcklund transformation [4,8]. With a suitable variable transformation and eliminating the auxiliary variable z, the bilinear equations (1.13)–(1.14) give to a differential-difference system which is a little complicated and we omit the concrete form here. However, the system is nonlocal about the discrete variable n and this makes it difficult to apply some numerical discretization on the temporal variable to design numerical algorithms. Different from system (1.13)–(1.14) or its nonlinear form, system (1.9)–(1.12) is continues in x and we succeed in discretizing it with Pseudo-spectral method and make it an effective numerical scheme.

As we analyzed, the discrete system (1.9)–(1.12) converges to the Boussinesq equation (1.2)–(1.3) when k tends to zero. However, in the case of $\delta < 0$, the discrete system is an integrable analogue to the 'Bad' Boussinesq equation and there is no such convergence relation between their solutions. We will discuss this in Section 3. As we know, the 'Bad' Boussinesq equation is linearly unstable and numerical computations are at best unreliable. This might be a good explanation to the misconvergence.

For the case of $\delta > 0$ or 'Good' Boussinesq equation, we make several numerical experiments with the discrete system (1.9)–(1.12) and Pseudo-spectral discretization of space variable. As we all know, there is no efficient technique to move out the phase shift caused by discretization when doing numerical simulations. Notice that we have two choices for a, positive and negative. And this gives us a way to reduce the phase shift phenomena. See details in Sections 3.1 and 4.2.

The paper is organized as follows. In Section 2, we derive a Bäcklund transformation and a Lax pair of the deferential-difference system. In Section 3, we give its soliton solutions via Hirota's bilinear method and make some analysis about its real solutions. In Section 4, we use the Pseudo-spectral methods to discrete the space variable and carry out several numerical experiments. The numerical results coincide with our analysis.

2. Integrability

In this section, we study the integrability of the differential-difference system (1.9)–(1.12) or its bilinear form (1.7)–(1.8). The results include a Bäcklund transformation and a Lax pair. We start from the bilinear representations. Actually, there is a systematic method to derive Bäcklund transformation for bilinear equations [4]. With a suitable variable transformation, a Lax pair can be derived from the Bäcklund transformation.

2.1. Bäcklund transformation

We have the following proposition

Proposition 2.1. *The bilinear system (1.7)–(1.8) has the Bäcklund transformation*

$$(D_s-aD_x^2)f_m\cdot g_m=0,\tag{2.1}$$

$$(aD_xD_s+\delta D_x^3-D_x)f_m\cdot g_m=\lambda f_m g_m,\tag{2.2}$$

$$(aD_s+3\delta D_x^2+(\frac{2a}{k}-\frac{1}{2}))e^{-\frac{1}{2}D_m}f_m\cdot g_m=\mu e^{\frac{1}{2}D_m}f_m\cdot g_m,\tag{2.3}$$

where λ and μ are arbitrary constants.

Proof. Let f_m be a solution of (1.7)–(1.8) and g_m be given by (2.1)–(2.3). Set

$$P_1\equiv[(D_s^2-D_x^2+\delta D_x^4)g_m\cdot g_m]f_m^2,\tag{2.4}$$

$$P_2\equiv[(aD_xD_s-\delta D_x^3-\frac{2a}{k}D_x)e^{\frac{1}{2}D_m}g_m\cdot g_m](e^{\frac{1}{2}D_m}f_m\cdot f_m).\tag{2.5}$$

If we can prove $P_1 = 0$, $P_2 = 0$, then g_m is also a solution of system (2.1)–(2.3) which means that equations (2.1)–(2.3) construct a