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Influence of stability islands in the recurrence of particles in a static oval billiard with holes

Matheus Hansen [a,*], R. Egydio de Carvalho [b], Edson D. Leonel [c,d]

[a] Instituto de Física da Universidade de São Paulo, Rua do Matão, Travessa R 187, Cidade Universitária, 05314-970 São Paulo, SP, Brazil
[b] Universidade Estadual Paulista – UNESP, Rio Claro, SP, Brazil
[c] Departamento de Física, UNESP – Univ Estadual Paulista, Av. 24A, 1515, Bela Vista, 13506-900, Rio Claro, SP, Brazil
[d] Abdus Salam International Center for Theoretical Physics, Strada Costiera 11, 34151 Trieste, Italy

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Statistical properties for the recurrence of particles in an oval billiard with a hole in the boundary are discussed. The hole is allowed to move in the boundary under two different types of motion: (i) counterclockwise periodic circulation with a fixed step length and; (ii) random movement around the boundary. After injecting an ensemble of particles through the hole we show that the surviving probability of the particles without recurring – without escaping – from the billiard is described by an exponential law and that the slope of the decay is proportional to the relative size of the hole. Since the phase space of the system exhibits islands of stability we show there are preferred regions of escaping in the polar angle, hence given a partial answer to an open problem: *Where to place a hole in order to maximize or minimize a suitable defined measure of escaping.*

1. Introduction

A billiard is a dynamical system where a point-like particle moves with constant speed along straight lines confined to a piecewise smooth boundary to where it experiences specular reflections [1]. In such type of collisions the tangent component of the velocity of the particle, measured with respect to the border where collision happened, is unchanged while the normal component reverses sign. Originally, the investigation on billiards was introduced in the seminal paper of Birkhoff [2] in the beginning of last century – therefore introducing a new research area – and since from there the scientific research on this topic has experienced a great development. Indeed, Birkhoff considered the investigation of the motion of a free point-like particle in a bounded manifold. Modern investigations on billiards however are connected with the results of Sinai [3] and Bunimovich [4,5] who made rigorous demonstrations in the subject. The billiards theory has also been used in many different kinds of physical systems, including experiments on superconductivity [6], wave guides [7], microwave billiards [8, 9], confinement of electrons in semiconductors by electric potentials [10,11], quantum tunneling [12], plasma physics [13,14] and many others.

The dynamics of a particle in a billiard can be matched into one of the following three possibilities: (i) regular; (ii) ergodic and; (iii) mixed. The circular billiard is a typical example of case (i) since it is integrable due to the conservation of energy and angular momentum [1]. The phase space is filled with straight lines. Other example of such a system is the elliptical billiard in which the energy and the angular momenta about the two foci are preserved [15]. Case (ii) corresponds to systems containing zero measure stable periodic orbits, hence dominated by chaotic dynamics, as the Bunimovich stadium [4,16] as well as the Sinai billiard [3]. Finally the case (iii) is the most common among them, and in such systems the phase space is composed by Kolmogorov–Arnold–Moser (KAM) islands surrounded by a chaotic sea which is limited by a set of invariant spanning curves [17,18]. Typical example includes the annular billiard [12]. In Ref. [15], Sir Michael Berry discussed a family of billiards of the oval-like shapes. The radius in polar coordinates has a control parameter, (ϵ), which leads to a smooth transition from a circumference with $(\epsilon = 0)$ – hence integrable – to a deformed form with $(\epsilon \neq 0)$. For sufficiently small (ϵ), a special set of invariant spanning curves exists in the phase space corresponding to the so-called whispering gallery orbits. They are orbits moving around the billiard, close to the border, with either positive (counterclockwise dynamics) or negative (clockwise dynamics) angular momentum. As soon as the parameter reaches a critical

* Corresponding author.
E-mail addresses: mathehansen@gmail.com (M. Hansen), regydio@rc.unesp.br (R. Egydio de Carvalho), edleonel@rc.unesp.br (E.D. Leonel).

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value [19], the invariant spanning curves are destroyed as well as the whispering gallery orbits.

Billiards can also be considered in the context of recurrence of particles [20,21], particularly related to the Poincaré recurrence [20,22]. The recurrence can be measured from the injection and hence from the escape of an ensemble of particles by a hole made in the boundary. The dynamics is made such that a particle injected through the hole is allowed to move inside the billiard suffering specular reflections with the boundary until it encounters the hole again. At this point the particle escapes from the billiard. The number of collisions that the particle has had till the escape is computed and another particle with different initial condition is introduced in the system. The dynamics is repeated until a large ensemble of particles is exhausted. The statistics of the recurrence time is then obtained. The known results are that for a totally chaotic dynamics, the survival probability – probability that the particle survives without escaping through the hole – is described by an exponential function [23,24]. However, if there are resonance islands, it is possible to observe a different behavior for the survival probability. For this situation, we can observe some chaotic orbits very close to islands of stability in such way that these orbits can be trapped around the islands for a long time influencing the survival probability. In this case, the survival probability is changed to a lower decay, as conjectured in Ref. [22], to be described by a power law of universal scaling with an exponent from the order of $\simeq -2.57$. Therefore, as considered recently in a book chapter by Dettmann [23] who discusses some open problems in billiard with holes, a particular question was posed regarding to escape of particles: *Optimisation: Specify where to place a hole to maximize or minimize a suitable defined measure of escaping.*

In the current paper we discuss the recurrence of particles in an oval-like shaped billiard with a hole in the boundary and our main goal is to move a step further as an attempt to give a partial answer to the above question. We know that if we have just one hole and we choose any position for it, some initial conditions injected through the hole will escape. The escaping time is fast for some positions of the hole. So, the idea to move the hole along the billiard boundary allows us to analyze the rate of escape, or the survival of the particles, in different positions through the escaping time. In our simulations the hole is allowed to move around the boundary under two different rules: (i) periodic and; (ii) random. In either cases, we define fixed places around the boundary to where the hole can be introduced. In the case (i) the hole moves counterclockwise under two circumstances. As soon as the particle is injected through the hole, its position moves if the particle escapes through it with less than 5 collisions with the boundary. If the particle does not escape until 5 collisions, it moves counterclockwise to a neighboring allowed position and waits until a escape or to more than other 5 collisions. The billiard perimeter is divided into 63 equal steps for the hole tour. This process repeats injecting and escaping particles until all the ensemble is exhausted. In the case (ii) the hole moves randomly around the boundary respecting the time of 5 collisions. The survival probability, obtained from the recurrence time that the particle spents to escape, is accounted for a large ensemble of noninteracting particles. At each time that the particle escapes, the polar angle and the angle of the trajectory of the particle are known from the equations of the map. Then a statistics of the density of particles that escaped from a given region of the phase space can be computed. We show that the density of escape measured in both polar angle as well as the angle of the particle's trajectory present peaks and valleys. The peaks are associated to the high density occupation in the phase space while the valleys are mostly linked to the periodic islands domain. Our results then give a partial answer to the above open question, at least for the oval billiard which has mixed phase space.

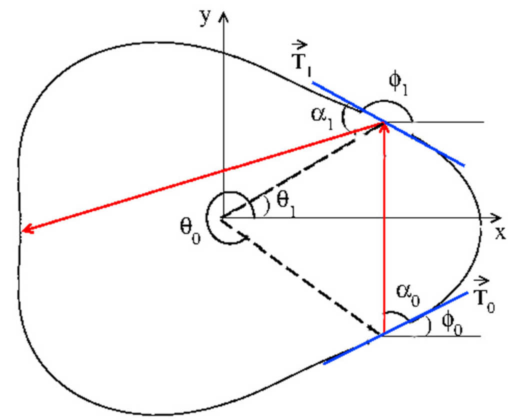

Fig. 1. (Color online.) Illustration of the angles involved in the billiard.

This paper is organized as follows. In Sec. 2 we discuss the model and the equations that fully describe the dynamics of the system. The escape properties for the particles when the hole moves periodically around the boundary are made in Sec. 3. The survival probability for the particles when the hole moves randomly around the boundary is discussed in Sec. 4 while our final remarks and conclusions are drawn in Sec. 5.

2. The static oval billiard

We discuss in this section how to obtain the equations that fully describe the dynamics of the system. To start with, the radius of the boundary in polar coordinate is given by

$$R(\theta, \epsilon, p) = 1 + \epsilon \cos(p\theta), \tag{1}$$

where θ is the polar coordinate, ϵ corresponds to a perturbation parameter of the circle and $p > 0$ is an integer number. For $\epsilon = 0$ the system is integrable. The phase space is foliated [1] and only periodic and quasi-periodic orbits are observed. For $\epsilon \neq 0$ the phase space is mixed containing both periodic, quasi-periodic and chaotic dynamics. When ϵ reaches the critical value [19] $\epsilon_c = 1/(1 + p^2)$ the invariant spanning curves, corresponding to the whispering gallery orbits are destroyed and only chaos and periodic islands are observed. This happens when the boundary is concave for $\epsilon < \epsilon_c$ and is not observed for $\epsilon > \epsilon_c$ when the boundary exhibits segments that are convex.

The dynamics is described by a two dimensional nonlinear mapping relating the variables $(\theta_n, \alpha_n) \to (\theta_{n+1}, \alpha_{n+1})$ where θ denotes the polar angle to where the particle collides and α represents the angle that the trajectory of the particle does with a tangent line at the collision point. Fig. 1 illustrates the representation of the angles.

For an initial condition (θ_n, α_n) the position of the particle is written as $X(\theta_n) = [1 + \epsilon \cos(p\theta_n)] \cos(\theta_n)$ and $Y(\theta_n) = [1 + \epsilon \cos(p\theta_n)] \sin(\theta_n)$. The angle of the tangent vector at the polar coordinate θ_n is $\phi_n = \arctan\left[\frac{Y'(\theta_n)}{X'(\theta_n)}\right]$, where $X'(\theta) = dX(\theta)/d\theta$ and $Y'(\theta) = dY(\theta)/d\theta$. Since there are no forces acting on the particle from collision to collision, it then moves along a straight line so its trajectory is given by

$$Y(\theta_{n+1}) - Y(\theta_n) = \tan(\alpha_n + \phi_n)[X(\theta_{n+1}) - X(\theta_n)], \tag{2}$$

where θ_{n+1} is the new polar coordinate of the particle when it hits the boundary, which is to be obtained numerically. The angle α_{n+1} given the slope of the trajectory of the particle after a collision is

$$\alpha_{n+1} = \phi_{n+1} - (\alpha_n + \phi_n). \tag{3}$$