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Coupled oscillators with parity-time symmetry

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Different models of coupled oscillators with parity-time (PT) symmetry are studied. Hamiltonian functions for two and three linear oscillators coupled via coordinates and accelerations are derived. Regions of stable dynamics for two coupled oscillators are obtained. It is found that in some cases, an increase of the gain-loss parameter can stabilize the system. A family of Hamiltonians for two coupled nonlinear oscillators with PT-symmetry is obtained. An extension to high-dimensional PT-symmetric systems is discussed.

1. Introduction

A set of coupled oscillators is a basic model of interacting systems. Many important ideas, such as the energy exchange, the eigenfrequency splitting and normal modes, are introduced in the study of this model, see e.g. [1,2].

Many types of linear coupled oscillators can described by a Hamiltonian, which is a quadratic function of coordinates and momenta. Energy is conserved in Hamiltonian systems, therefore conservative systems are associated usually with *ideal* systems without dissipation (loss) and amplification (gain). Let us consider a set of two coupled oscillators:

$$\ddot{x}_1 + 2\gamma\dot{x}_1 + \omega_0^2 x_1 + \kappa x_2 = 0,$$
$$\ddot{x}_2 - 2\gamma\dot{x}_2 + \omega_0^2 x_2 + \kappa x_1 = 0 \quad (1)$$

where x_1 and x_2 are the coordinates of the oscillators, γ is the parameter of dissipation (for x_1) and amplification (for x_2), ω_0 is the frequency of a single oscillator, and κ is the coupling parameter. The overdot denotes the derivative on t. Model (1) represents an open system with energy flow from the second oscillator (assuming $\gamma > 0$) to the first oscillator. System (1) has been studied in Refs. [3–6]. Surprisingly, it was found in Ref. [5] that system (1) has the corresponding Hamiltonian.

Model (1) is a simple example of systems with parity-time (PT) symmetry [7,8]. Basically, this means that the system is invariant under inversion of both space and time. If one interchanges x_1 and x_2, and change t to $-t$, Eqs. (1) remains the same. Usually, PT-symmetric systems are stable for some range of the system parameters, and they become unstable when a certain parameter exceed the symmetry breaking threshold [7].

A notion of PT-symmetry came from attempts to extend quantum mechanics beyond Hermitian operators [9]. A typical PT-symmetric Hamiltonian has a complex-valued potential $U(\mathbf{r})$. The imaginary part of $U(\mathbf{r})$ characterizes amplification and dissipation of the wave function. Therefore, a PT-symmetric quantum system is a model with distributed gain and loss. When gain and loss are well balanced, the system is in a stationary state. Later, this idea was expanded to other fields of physics, such as classical mechanics, electronics, and optics. The idea is very promising in optics, where a number of interesting applications has been realized. These include the double refraction, unidirectional light propagation [10–13], perfect absorbers [14] and lasers [15,16].

In present paper, firstly, we present various physical systems that obey PT-symmetry. Then, we consider an extension of model (1), which is Hamiltonian as well. Moreover, we find nonlinear generalization of the model, which is also Hamiltonian, similar to that analyzed in Ref. [17]. Systems of three and more degrees of freedom are also discussed.

2. PT-symmetric models

In this Section, we present several models with PT-symmetry. The aim of this Section is to demonstrate that a PT-symmetric model is not an abstract notion, but it can describe real physical systems. A PT-symmetric system requires an element that provide amplification, or negative dissipation. Such elements are discussed, for example, in Ref. [1], and we use them to construct different types of PT-symmetric systems.

We start with two coupled oscillators shown in Fig. 1(a). Two masses, m_1 and m_2, are connected with each others and walls by

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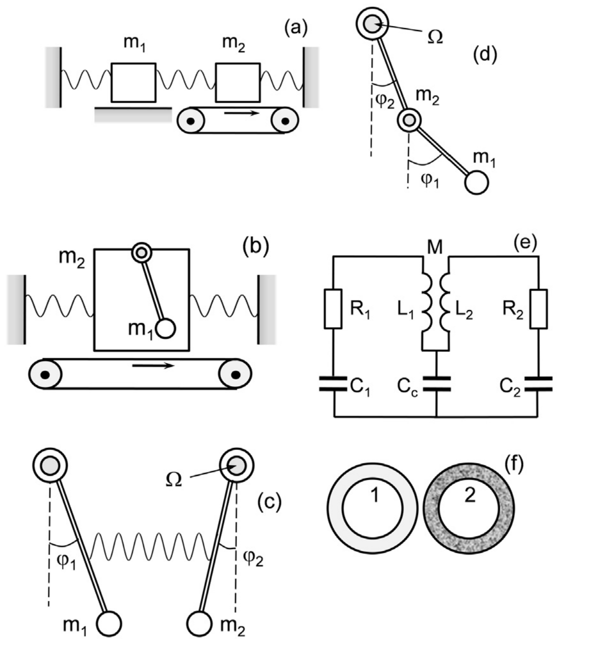

Fig. 1. Examples of PT-symmetric models: (a–d) mechanical systems, (e) an electrical system, and (f) coupled optical waveguides, where the right (darker) waveguide has resonance atoms.

means of springs. Mass m_1 is placed on a fixed frictionless surface, however there is dissipation of energy due to surrounding media. We assume that the dissipation force is proportional to the oscillator velocity. The second mass is placed on a conveyor, which moves with constant velocity V_c. The conveyor drags the mass because of friction. In the absence of coupling between the masses, the equation of motion of m_2 is the following [1]:

$$\ddot{x}_2 + \Gamma \dot{x}_2 + \omega_0^2 x_2 = F(\dot{x}_2 - V_c), \tag{2}$$

where F is the force that depends on the relative velocity of the body and the conveyor. For small velocities $\dot{x}_2$, one can expand $F \approx F(V_c) + F'(-V_c)\dot{x}_2$. A constant force $F(V_c)$ results in a shift of the stationary point for x_2, while the second term results in modification of the dissipation parameter. By a proper choice of $F'(-V_c)$, one can make the dissipation parameter negative, $\Gamma - F'(-V_c) = -2\gamma$, which results in amplification. Then, with a corresponding choice of the system parameters, the model in Fig. 1(a) is reduced to Eqs. (1).

Oscillators in Fig. 1(a) are coupled via coordinates x_1 and x_2. However, it is possible to make inertial coupling between oscillators as shown in Fig. 1(b). Mass m_1 oscillates with dissipation inside m_2. Amplification for mass m_2 is achieved by means of a conveyor, as in Fig. 1(a). One can show that oscillators in this case are coupled via acceleration. (Actually, the equations of motion can be transformed further such that only coordinate coupling remains, however, the initial equations, derived from the Lagrangian, have coupling via acceleration, see e.g. Ref. [2].)

A PT-symmetric mechanical system can be realized with a set of two pendulums coupled via a spring, see Fig. 1(c). Each pendulum has a sleeve on the upper end of the rod. This sleeve is put on a shaft (gray circles in Fig. 1(c)). The shaft of the second pendulum rotates with constant angular frequency Ω. The rotating shaft, similar to the moving conveyor in Figs. 1(a) and (b), introduces amplification for the second pendulum. We mention that a single pendulum with rotating shaft is called the Froude pendulum, see e.g. [1]. In linear approximation, the dynamics of pendulums is described by Eqs. (1). Oscillators in Fig. 1(c) are coupled via coordinates ϕ_1 and ϕ_2, similar to those in Fig. 1(a).

It is possible also to introduce acceleration coupling between the pendulums, as in a double pendulum presented in Fig. 1(d). The first pendulum in Fig. 1(d) is attached to the second one via a movable joint. The second pendulum is the Froude pendulum, so that the rotating shaft provides amplification.

The next example is a pair of oscillatory circuits, presented Fig. 1(e). Two RLC circuits are coupled with each other via mutual inductance M (acceleration coupling) and capacitor C_c (coordinate coupling). The main difference of this circuit from conventional ones is that R_2 has negative resistance, providing gain in the system. Negative resistance can be realized using a tunnel diode or an operational amplifier. PT-symmetric electronic circuits have been studied in Refs. [3,4]. Also, a system of two Josephson junctions with capacitive coupling is modeled by two pendulums coupled via acceleration [18]. Then, it is possible to realize a PT-symmetric system in such superconducting circuits by implementing the negative resistance in either junction.

The last example in this Section is a system of two circular optical waveguides in Fig. 1(f). The waveguides of a size of few microns and less are coupled due to interaction of evanescent fields, therefore it is a coordinate coupling. The second waveguide has resonance atoms inside that can be pumped by external light. This creates amplification in the waveguide, so that with a proper choice of parameters, the system can be considered as PT-symmetric. Similar systems are studied, for example, in Refs. [5,15,16]. The oscillation of electromagnetic fields inside the waveguides is described in linear approximation by equations similar to Eqs. (1).

It is interesting also to consider the quantum behavior of coupled PT-symmetric oscillators. Several examples are presented in Refs. [5,19], see also reviews [7,8]. In Ref. [5], the twofold bifurcation is observed in the classical and quantized versions of the system. In Ref. [19], a relation between a symmetric quadratic Hamiltonian (c.f. Sec. 3) and a pseudo-Hermitian matrix is obtained. We expect that coupling via acceleration can add new features to the dynamics of quantum oscillators.

The examples presented in this Section shows that two types of coupling exist, namely via coordinates and via accelerations. In the next Section, we obtain a model that include both types of coupling.

3. Hamiltonian systems of PT-symmetric oscillators

3.1. Two linear oscillators

In this Section we derive an extension of system (1), which is also Hamiltonian. We start with a general expression for the Hamiltonian written as a quadratic form:

$$H = \mathbf{z}^T \mathbf{A} \mathbf{z}, \tag{3}$$

where $\mathbf{A} = \{a_{ij},\ i, j = 1, \ldots, 4\}$ is a 4×4-matrix, $\mathbf{z} = (x_1, x_2, p_1, p_2)^T$ is a column vector, p_1 and p_2 are canonical momenta of coordinates x_1 and x_2, respectively, and superscript T denote transposition. Without loss of generality, one can assume that $\mathbf{A}$ is symmetric, since a quadratic form with an anti-symmetric matrix equals zero.

The equations of motion are obtained from the Hamilton equations:

$$\dot{x}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial x_k}, \quad k = 1, 2. \tag{4}$$

From equations for $\dot{x}_k$, we find the relation between momenta and velocities. Then, equations for $\dot{p}_k$ give the following equations of motion