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Correction of harmonic motion and Kepler orbit based on the minimal momentum uncertainty

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ABSTRACT

In this paper we consider the deformed Heisenberg uncertainty principle with the minimal uncertainty in momentum which is called a minimal momentum uncertainty principle (MMUP). We consider MMUP in D-dimension and its classical analogue. Using these we investigate the MMUP effect for the harmonic motion and Kepler orbit.

1. Introduction

The ordinary Heisenberg uncertainty principle $\Delta x \Delta p \geq \hbar/2$ does not explain the existence of a minimum measurable length because Δx goes to zero in the high momentum limit. Thus, to incorporate the concept of minimum measurable length into quantum mechanics, one should deform the ordinary Heisenberg uncertainty principle which is called Generalized Uncertainty Principle (GUP) whose commutation relation is given by $[X, P] = i\hbar(1 + \beta P^2)$. In the last decade, many papers have been appeared in the literature to address the effects of GUP on the quantum mechanical systems especially in high energy regime [1–16].

There exists another possibility to deform Heisenberg uncertainty principle so that it possesses the minimal uncertainty in momentum. It is well known that in an (anti) de Sitter background the Heisenberg uncertainty principle should be modified by introducing corrections proportional to the cosmological constant $\Lambda = -3/l_H^2$ with l_H the (anti) de Sitter radius [17], where $l_H^2 < 0$ for de Sitter spacetime, and $l_H^2 > 0$ for anti de Sitter spacetime:

$$\Delta X_i \Delta P_i \geq \frac{\hbar}{2}\delta_{ij}\left[1 + \frac{(\Delta X_i)^2}{l_H^2}\right] \tag{1}$$

This modification of the Heisenberg relation was named extended uncertainty principle (EUP). It has been motivated either by analogy with the GUP, or by gedanken experiments in which the expansion of the universe during a measurement is taken into account [8]. More recently, it has been shown by Mignemi [18] that it can also be derived from the definition of quantum mechanics on a de Sitter background, with a suitably chosen parametrization. He found that the relation (1) could be derived from the geometric properties of (anti) de Sitter spacetime. He also found the modified Heisenberg algebra corresponding to the EUP (1) as

$$[X_\mu, X_\nu] = 0, \ \ [P_\mu, P_\nu] = i\hbar\frac{1}{l_H^2}L_{\mu\nu}$$
$$[X_\mu, P_\nu] = i\hbar\left(\eta_{\mu\nu} + \frac{1}{l_H^2}X_\mu X_\nu\right) \tag{2}$$

where $\mu, \nu = 0, 1, 2, 3$ and $L_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu$.

As another deformation of the ordinary uncertainty relation, the symmetric generalized uncertainty principle (SGUP) appeared in some literatures [17,19–21] where the deformed uncertainty relation takes the following form:

$$\Delta X \Delta P \geq \hbar\left(1 + \frac{(\Delta X)^2}{L^2} + \ell^2\frac{(\Delta P)^2}{\hbar^2}\right) \tag{3}$$

Here, L is another uncertainty constant and the cutoff ℓ may be chosen as a string scale in the context of the perturbative string

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theory or Plank scale based on the quantum gravity. The thermodynamic quantities and the stability of a black hole in a cavity using the Euclidean action formalism by Gibbons and Hawking based on the SGUP were studied by some authors [22].

In this paper we consider the more general deformed Heisenberg uncertainty principle whose special case is given as the eq. (2). From now on we will call this deformed uncertainty principle a minimal momentum uncertainty principle (MMUP). As classical analogue of the MMUP, we introduce the MMUP corrected Poisson bracket. Using it, we obtain the MMUP corrected equation of motion and investigate the MMUP effect for the harmonic motion and Kepler orbit.

2. D-dimensional MMUP

Now let us start with the D-dimensional MMUP corrected Heisenberg algebra:

$$[X_i, P_j] = i\hbar[\delta_{ij} f(R) + g(R)X_iX_j], \quad [X_i, X_j] = 0$$
$$[P_i, P_j] = i\hbar h(R)[X_iP_j - X_jP_i] \tag{4}$$

where $i, j = 1, 2, \cdots, D$, $R = \sqrt{\sum_{i=1}^{D} X_i^2}$, and $f(R), g(R)$ and $h(R)$ are functions in R^2. From the Jacobi identity

$$[[P_i, P_j], X_k] + [[P_j, X_k], P_i] + [[X_k, P_i], P_j] = 0 \tag{5}$$

we have the following relation for $f(R), g(R)$ and $h(R)$:

$$hf = fg - rgf' - \frac{1}{r} ff' \tag{6}$$

where $'$ implies the derivative with respect to R. The position and momentum operators can be represented by

$$P_i = \frac{\hbar}{i}[f(r)\partial_i + g(r)x_ix_j\partial_j], \quad X_i = x_i \tag{7}$$

where the operators x_i and p_i satisfy the canonical commutation relations $[x_i, p_j] = i\hbar\delta_{ij}$. If we take $f(r) = 1 + \alpha R^2$, $g(r) = \alpha'$, we have

$$[X_i, P_j] = i\hbar[\delta_{ij}(1 + \alpha R^2) + \alpha' X_iX_j], \quad i, j = 1, 2, \cdots, D$$
$$[X_i, X_j] = 0$$
$$[P_i, P_j] = i\hbar\frac{\alpha' - 2\alpha - \alpha(\alpha' + 2\alpha)R^2}{1 + \alpha R^2}[X_iP_j - X_jP_i] \tag{8}$$

3. Classical mechanics based on MMUP

Let us investigate the classical limit of the algebra (8). Recall that the quantum mechanical commutator corresponds to the Poisson bracket in classical mechanics via

$$\frac{1}{i\hbar}[A, B] \to \{A, B\} \tag{9}$$

The MMUP-corrected Poisson bracket of two functions $F(x_1, \cdots, x_D, p_1, \cdots, p_D), G(x_1, \cdots, x_D, p_1, \cdots, p_D)$ in D dimensions is defined as

$$\{F, G\} = \sum_{i,j}[\delta_{ij}(1 + \alpha r^2) + \alpha' x_ix_j]\left(\frac{\partial F}{\partial x_i}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial x_i}\right)$$
$$+ \sum_{i,j}[\alpha' - 2\alpha - \alpha(\alpha' + 2\alpha)r^2]L_{ij}\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial p_j} \tag{10}$$

where the deformed angular momentum is defined as

$$L_{ij} = \frac{x_ip_j - x_jp_i}{1 + \alpha r^2} \tag{11}$$

In particular, we find that the time evolutions of the coordinates and momenta are governed by

$$\dot{x}_i = \{x_i, H\} \tag{12}$$

$$\dot{p}_i = \{p_i, H\} \tag{13}$$

Now let us consider the Hamiltonian of a particle in a central force potential

$$H = \frac{1}{2m}p^2 + V(r) \tag{14}$$

where $r = \sqrt{\sum_{i=1}^{D} x_ix_i}$. The time evolutions of the coordinates and momenta in this case are

$$\dot{x}_i = \frac{p_i}{m}(1 + \alpha r^2) + \frac{\alpha'}{m}x_ix_jp_j \tag{15}$$

$$\dot{p}_i = -\left(\frac{1 + (\alpha + \alpha')r^2}{r}\right)V'(r)x_i$$
$$+ \frac{1}{m}[\alpha' - 2\alpha - \alpha(\alpha' + 2\alpha)r^2]L_{ij}p_j \tag{16}$$

where the summation convention is used. Here deformed angular momentums are generators of rotation:

$$\{x_k, L_{ij}\} = x_i\delta_{kj} - x_j\delta_{ki}, \quad \{p_k, L_{ij}\} = p_i\delta_{kj} - p_j\delta_{ki} \tag{17}$$

For motion in a central force potential, L_{ij} is conserved due to rotational symmetry:

$$\{L_{ij}, H\} = 0 \tag{18}$$

So we have

$$L^2 = -\frac{1}{2}L_{ij}L_{ji} = \frac{r^2p^2 - (\vec{r}\cdot\vec{p})^2}{(1 + \alpha r^2)^2} \tag{19}$$

The conservation of the deformed angular momentum shows that the motion of the particle will be confined to a 2-dimensional plane. Thus, we can consider the situation where the motion is in the x_1x_2-plane, which implies that the non-vanishing components of the deformed angular momentum are $L_{12} = -L_{21} = l$. Then, the motion of a particle is described by r and ϕ defined as

$$r = \sqrt{x_1^2 + x_2^2}, \quad \phi = \tan^{-1}\frac{x_2}{x_1} \tag{20}$$

The equation of motion for r is then given by

$$\dot{r} = \frac{1}{m}[1 + (\alpha + \alpha')r^2]p_r \tag{21}$$

where

$$p_r = \frac{(\vec{r}\cdot\vec{p})}{r} = \sqrt{p^2 - \frac{l^2}{r^2}(1 + \alpha r^2)^2} \tag{22}$$

Since the energy E is also conserved, we can rewrite the eq. (21) as

$$\dot{r} = \frac{1}{m}[1 + (\alpha + \alpha')r^2]\sqrt{2m(E - V(r)) - \frac{l^2}{r^2}(1 + \alpha r^2)^2} \tag{23}$$

The equation of motion for the angle ϕ is

$$\dot{\phi} = \frac{l}{mr^2}(1 + \alpha r^2)^2 \tag{24}$$

With a help of the eq. (23) and the eq. (24), we get

$$\frac{d\phi}{dr} = \frac{l}{r^2}\frac{(1 + \alpha r^2)^2}{(1 + (\alpha + \alpha')r^2)\sqrt{2m(E - V(r)) - \frac{l^2}{r^2}(1 + \alpha r^2)^2}} \tag{25}$$