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Spatial statistics of magnetic field in two-dimensional chaotic flow in the resistive growth stage

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ABSTRACT

The correlation tensors of magnetic field in a two-dimensional chaotic flow of conducting fluid are studied. It is shown that there is a stage of resistive evolution where the field correlators grow exponentially with time. The two- and four-point field correlation tensors are computed explicitly in this stage in the framework of Batchelor–Kraichnan–Kazantsev model. They demonstrate strong temporal intermittency of the field fluctuations and high level of non-Gaussianity in spatial field distribution.

1. Introduction

Kinematic dynamo consists in enhancement of magnetic field fluctuations in a moving conducting fluid. This enhancement is statistically significant for non-stationary flows, in particular, chaotic flows. The dynamo is relevant for astrophysics [1–5] and for so-called elastic turbulence which is the chaotic motion of polymer solutions [6–8]. In both cases the velocity field $\boldsymbol{v}(\boldsymbol{r},t)$ can be considered as a smooth function of coordinates $\boldsymbol{r}$ allowing the use of the Taylor expansion. Practically this means that in the Lagrangian frame comoving with a given liquid particle the profile of the field $\boldsymbol{v}(\boldsymbol{r},t)$ is approximately linear:

$$v_\alpha(\boldsymbol{r},t) \approx \sigma_{\alpha\beta}(t) r_\beta \tag{1}$$

with matrix-valued random function $\sigma_{\alpha\beta}(t)$ of time t. This time-dependent linear profile can be used for $r \ll R_v$ where R_v is the characteristic scale of the flow, e.g., the size of the space domain occupied by the flow. Evolution of the magnetic field $\boldsymbol{B}(\boldsymbol{r},t)$ is governed by the equation [9]:

$$\partial_t \boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{B} + \kappa\nabla^2\boldsymbol{B}, \tag{2}$$

where κ is the dissipation coefficient inverse proportional to the conductivity of the fluid.

The present paper is devoted to computation of correlation tensors of the field $\boldsymbol{B}$ for two-dimensional incompressible chaotic flow at conditions when the expansion (1) is applicable. The statistics of the traceless matrix $\sigma_{\alpha\beta}(t)$ is assumed to be the Gaussian δ-correlated in time random process:

$$\langle \sigma_{\alpha\mu}(t)\sigma_{\beta\nu}(t')\rangle = \mathcal{G}\left(3\delta_{\alpha\beta}\delta_{\mu\nu} - \delta_{\beta\nu}\delta_{\alpha\mu} - \delta_{\beta\mu}\delta_{\alpha\nu}\right)\delta(t-t'), \tag{3}$$

where the Stratonovich regularization of the δ-function is assumed, i.e., the δ-function is treated as the limit of a sequence of symmetric with respect to reflection $(t-t') \to (t'-t)$ functions. It should be stressed that this is an external requirement and important part of the model formulation. Such tensor structure follows from the expansion of the velocity field $\boldsymbol{v}(\boldsymbol{r},t)$ correlator in Eulerian frame:

$$\langle v_\mu(\boldsymbol{r},t)v_\nu(\boldsymbol{0},t')\rangle \approx \left[V_0^2\delta_{\mu\nu} - \mathcal{G}\left(\frac{3}{2}r^2\delta_{\mu\nu} - r_\mu r_\nu\right)\right]\delta(t-t'), \quad r \ll R_v. \tag{4}$$

We assume that for the case of spatially homogeneous statistics the characteristic scale R_v coincides with the correlation length of the flow. We consider here the case when $R_v \gg r_d = 2\sqrt{\kappa/\mathcal{G}}$ which is suitable for astrophysical and rheological applications.

It was shown [5,10,11] for three-dimensional chaotic flows described by this BKK (Batchelor–Kraichnan–Kazantsev) model that the moments $\langle \boldsymbol{B}^{2n}(\boldsymbol{r},t)\rangle$, $n = 1, 2, \dots$ grow exponentially with time and the correlation functions of the field tend to universal spatial shapes. These shapes correspond to filaments with increasing length in which the magnetic field is concentrating in the course

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of evolution. The width of such filaments is decreasing up to the dissipative scale r_d where the value if the width is stabilized. At this moment the resistive stage of the evolution begins. The exponential growth continues but the exponent changes (see also review [12]).

Two-dimensional flows were not considered long time in dynamo studies since it was asserted in [14–16] that in this case the initial growth of the field changes to exponential decay. It should be emphasized that the magnetic field $\boldsymbol{B}(\boldsymbol{r},t)$ is considered to be three-dimensional both as a vector and as a function of space position, only the flow is two-dimensional: $\boldsymbol{v} = (v_1, v_2, 0)$. The absence of dynamo at largest times has been subsequently proved rigorously for flows with finite energy (see [19] and the book [20]). On the other hand, the implicit mathematical constructions of these papers as well as the qualitative arguments used in the physical works [14–16] don't allow to make a definite statement about the duration of the growth stage.

It was noted in [12,13,17] that the exponential increase of the magnetic field continues in the resistive regime when the magnetic diffusion is important. The two-dimensional flows considered in these papers have spatially homogeneous statistics similar to the model studied here. The growth of the field amplitude is accompanied by elongation of the field filaments like in the three-dimensional case [5,11]. Such behavior comes to a halt when the characteristic length of the filaments reaches R_v. Then the two-dimensionality of the flow changes the picture drastically producing anticorrelation of the magnetic field in neighboring filaments terminating the dynamo in agreement with exact mathematical theorems [19,20] related to infinite time limit. In [18] it is shown that the maximal value of $\langle \boldsymbol{B}^2\rangle$ reached in the course of evolution is $\sim \langle {\boldsymbol{B}_0}^2\rangle (R_c/r_d)^2 \gg \langle {\boldsymbol{B}_0}^2\rangle$ where $\boldsymbol{B}_0$ is the initial field amplitude. This enhancement is picked up in the exponential growth stage. Depending on initial level of fluctuations the growth stage may be terminated first by the back reaction of the magnetic field on the fluid flow [5,21]. In this case the spatial structure of the field fluctuations is extremely important. We show in the present paper that the spatial statistics of the magnetic field in the resistive growth regime in the framework of two-dimensional BKK-model is intermittent similar to three-dimensional case [11,12]. The path-integral formalism created to study passive scalar statistics [22,23] and isotropic correlations in three-dimensional dynamo problem [11] is developed here to compute the tensor structure of two- and four-point magnetic field correlators.

The evolution equation (2) conserves the divergence of the field $\boldsymbol{B}(\boldsymbol{r},t)$. On the other hand, the evolution equations for the magnetic field components in the flow plane $B_\mu(\boldsymbol{r},t)$, $\mu = 1,2$ and for the component $B_3(\boldsymbol{r},t)$ perpendicular to this plane decouple. This means that we can consider evolution of two-component divergence-full field $B_\mu(\boldsymbol{r},t)$, $\mu = 1,2$ separately since the condition

$$\partial_\mu B_\mu + \partial_3 B_3 = 0 \tag{5}$$

is satisfied by the component $B_3(\boldsymbol{r},t)$ governed be the decoupled evolution equation. The dependence of $B_\mu(\boldsymbol{r},t)$, $\mu = 1,2$ on the coordinate r_3, being essential in the condition (5), is unimportant in the evolution. The coordinate r_3 enters the correlation functions of components $B_\mu(\boldsymbol{r},t)$, $\mu = 1,2$ as a parameter only and we don't take care here on it.

2. Dynamical computation of the correlation tensors of the field $\boldsymbol{B_\alpha(r,t)}$

The change of the frame to (1) does not change the simultaneous statistics of the field $B_\mu(\boldsymbol{r},t)$, so that we use the following evolution equation:

$$\partial_t B_\alpha = \sigma_{\alpha\mu} B_\mu - \sigma_{\mu\nu} r_\nu \partial_\mu B_\alpha + \kappa \nabla^2 B_\alpha. \tag{6}$$

Performing the spatial Fourier transform:

$$B_\alpha(\boldsymbol{r},t) = \int \frac{d^2\mathbf{k}}{(2\pi)^2} e^{i\mathbf{kr}} b_\alpha(\boldsymbol{k},t) \tag{7}$$

we obtain for $b_\alpha(\boldsymbol{k},t)$ the first-order partial differential equation:

$$\partial_t b_\alpha = \sigma_{\alpha\mu} b_\mu + \sigma_{\mu\nu} k_\mu \frac{\partial}{\partial k_\nu} b_\alpha - \kappa k^2 b_\alpha. \tag{8}$$

Its solution with the initial data $\boldsymbol{b}(\boldsymbol{k},0)$ has the form:

$$b_\alpha(\boldsymbol{k},t) = (\hat W)_{\alpha\beta}(t)\, b_\beta\left(\hat W^T(t)\,\mathbf{k},0\right) \times \exp\left[-\kappa \int_0^t d\tau \left(\mathbf{k}\hat W(t,\tau)\hat W^T(t,\tau)\mathbf{k}\right)\right], \tag{9}$$

where the matrices $\hat W(t)$ and $\hat W(t,\tau)$ obey the equation

$$d\hat W/dt = \hat\sigma \hat W \tag{10}$$

and can be written as the ordered exponentials:

$$\hat W(t) = \mathrm{T}\exp\left(\int_0^t dt'\,\hat\sigma(t')\right),$$

$$\hat W(t,\tau) = \mathrm{T}\exp\left(\int_\tau^t dt'\,\hat\sigma(t')\right) = \hat W(t)\hat W^{-1}(\tau). \tag{11}$$

The incompressibility of the flow $\sigma_{\alpha\alpha} = 0$ leads to the unimodularity of the matrix $\hat W(t)$: $\det \hat W(t) = 1$.

To perform the averaging over the matrix Gaussian random process $\hat\sigma(t)$ we use the path integral formalism. The measure corresponding to the correlation function (3) has the form:

$$\mathcal{D}\hat\sigma(\tau)\exp\left\{-\frac{1}{16\mathcal{G}}\int_0^t d\tau\, L\left[\hat\sigma(\tau)\right]\right\},$$

$$L\left[\hat\sigma(\tau)\right] = 3\mathrm{Tr}\left(\hat\sigma\hat\sigma^T\right) + \mathrm{Tr}\left(\hat\sigma^2\right). \tag{12}$$

The matrix $\hat W(t)$ cannot be expressed as a functional of $\hat\sigma(t)$ explicitly. However, for the our problem this difficulty can be avoided (see also [23]). Let us perform the Iwasava parametrization of the matrix $\hat W(t)$:

$$\hat W = \hat{\mathcal{O}}(\varphi)\hat D(\rho)\hat T(\chi), \tag{13}$$

where

$$\hat{\mathcal{O}}(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}, \quad \hat D(\rho) = \begin{pmatrix} e^\rho & 0 \\ 0 & e^{-\rho} \end{pmatrix},$$

$$\hat T(\chi) = \begin{pmatrix} 1 & \chi(t) \\ 0 & 1 \end{pmatrix}, \tag{14}$$

and the parameters $\varphi(t)$, $\rho(t)$ and $\chi(t)$ are some functions of time t determined by $\hat\sigma(t)$ implicitly via the equation:

$$\hat\sigma(t) = d\hat W(t)/dt\,\hat W^{-1}(t) = \hat{\mathcal{O}}(\varphi)\begin{pmatrix} \dot\rho & \dot\varphi + \dot\chi e^{2\rho} \\ -\dot\varphi & -\dot\rho \end{pmatrix}\hat{\mathcal{O}}^{-1}(\varphi). \tag{15}$$

The initial conditions $\rho(0) = \chi(0) = \varphi(0) = 0$ correspond to the evident equality $\hat W(0) = 1$. If we consider the relation (15) as the change of variables in the measure (12) we obtain an explicit path