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Gossip algorithms in quantum networks

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ABSTRACT

Gossip algorithms is a common term to describe protocols for unreliable information dissemination in natural networks, which are not optimally designed for efficient communication between network entities. We consider application of gossip algorithms to quantum networks and show that any quantum network can be updated to optimal configuration with local operations and classical communication. This allows to speed-up – in the best case exponentially – the quantum information dissemination. Irrespective of the initial configuration of the quantum network, the update requiters at most polynomial number of local operations and classical communication.

1. Introduction

Real-world networks are complex: natural social and brain networks as well as artificial technological and computer networks exhibit non-trivial structural features, which make complete simulation of the network dynamics practically impossible [1]. Complex non-stationary structure of modern artificial networks becomes a serious obstacle in the design of optimal protocols for information dissemination in such networks. Inspired by a natural way of rumor spreading in social networks, gossip algorithms [2] give a simple strategy for distributed and robust information dissemination in a network of unknown structure. These algorithms have found prominent applications in sensor, peer-to-peer and social networks.

Quantum networks [3] will be the next generation of complex structures for communication and advanced information processing [4]. Due to quantum superposition and nonlocality [5], quantum networks exhibit a number of structural and dynamical features that classical networks lack. Among those are teleportation [6], quantum walks [7] and entanglement percolation [8,9] to name just a few. Recently we showed that with local operations and classical communication (LOCC) [5] one may change connectivity of a given quantum network and simulate complex entanglement graphs on a simple underlying quantum network [10]. The structural modifications may radically improve the network capacity for information dissemination and performance of corresponding protocols, such as gossip algorithms.

In this paper we consider the problem of optimal information dissemination in quantum networks and analyze performance of gossip algorithms on the networks. As intuition suggests, the network where any pair of vertices is connected with an edge offers the most favorable conditions for information dissemination. Such a network is represented with a complete graph. We show that any quantum network represented with a connected graph, i.e. where any two vertices can be connected with a path of edges, may be updated to the complete graph using just polynomial number of LOCC. The update allows to dissimilate information by means of quantum teleportation [5], thus radically improving the performance of the gossip algorithms on quantum networks.

This work is structured as follows. In the next section, we briefly describe classical gossip algorithms for single- and multi-piece information dissemination and introduce the quantities of interest, such as conductance, k-conductance and ε-dissemination time. For a more detailed and mathematically rigorous treatment we suggest an excellent review by Shah [2]. In Section 3, we show how to improve the performance of gossip algorithms on quantum networks by LOCC. For sparse quantum networks the improvement in the information dissemination time due to the update is exponential, but still requires only polynomial number of LOCC. We conclude in Section 4.

2. Classical gossip algorithms

From the structural viewpoint a network is a graph $G = (V, E)$ defined by sets of its vertices V and edges E. The set $V = \{1, \dots, n\}$

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consists of a finite countable number of n vertices. The edges represent connections between the vertices $E \subset V \times V$. The graph is called undirected if for any $(i, j) \subset E$, $(j, i) \subset E$ is also true. Here we impose no constrains on the direction of information dissimilation, hence consider only undirected graphs.

Information dissimilation on a graph may be studied with a discrete random walk technique, which requires definition of a $n \times n$ non-negative valued probability transition matrix $P = [P_{ij}]$, where P_{ij} is the probability of information dissemination from vertex i to j. Through the transition matrix, we may define an auxiliary function named conductance $\Phi(P)$, which characterizes the information dissemination capacity of a graph of particular configuration of vertices and edges. For symmetric P – which is the case for undirected graphs – the conductance is defined as [2]

$$\Phi(P) = \min_{S \subset V : |S| \leq n/2} \frac{\sum_{i \subset S,\, j \subset S^c} P_{ij}}{|S|}, \tag{1}$$

where S is the set of nodes that possesses the information, while S^c is the set of those that doesn't. Here, the minimum is to be found over the set S taking into account its symmetry. Hence, Eq. (1) is a proper definition of conductance for an undirected graph only [2]. The conductance is completely defined by the transition matrix of a graph, thus tells us how easy the information can be conducted through the graph. Also, the conductance is independent on a particular information dissemination protocol to be implemented on the graph.

A related to the conductance auxiliary function is k-conductance, which minimizes (1) for $k \leq n/2$, i.e.

$$\Phi_k(P) = \min_{S \subset V : |S| \leq k} \frac{\sum_{i \subset S,\, j \subset S^c} P_{ij}}{|S|}. \tag{2}$$

Using the k-conductance, we may also define the mean conductance $\hat{\Phi}(P)$ as

$$\hat{\Phi}(P) = \sum_{k=1}^{n-1} \frac{k}{\Phi_k(P)}. \tag{3}$$

In the following we will focus on two particular graphs: the complete graph, where each pair of nodes is connected with an edge, and the ring graph, where nodes are placed on a circle with edges between nearest neighbors only. These two graphs are chosen for comparison because of their radical difference in the capacity for information dissemination. With the probability matrix $P_{ij} = 1/n$ for all i and j, the complete graph has the best possible capacity to disseminate information, i.e. $\Phi(P) = O(1)$ and $\hat{\Phi}(P) = O(n^2 \log n)$, where $O(..)$ is the standard notation for asymptotic upper bound. The ring graph with the probability matrix $P_{ii} = 1/2$ and $P_{ij} = 1/4$ for $i \neq j$, in contrast, has the strongest constrain for information dissemination leading to $\Phi(P) = O(1/n)$ and $\hat{\Phi}(P) = O(n^3)$.

Analyzing gossip algorithms we will be interested in the value called ε-dissemination time $T(\varepsilon)$. This value gives us time by which all nodes have the information with probability at least $1 - \varepsilon$. The definition of the ε-dissemination time depends on the algorithm, thus will be given in the next sections for single- and multi-piece dissemination strategies separately. Our goal is to estimate the ε-dissemination time through the conductance, allowing general treatment of the algorithm efficiency for any graph structure.

2.1. Single-piece dissemination

Let an arbitrary vertex $\upsilon \in V$ have a piece of information that it wishes to spread to all the other vertices as quickly as possible. Let $S(t) \subset V$ denote the set of vertices that have the information at time t, which is also assumed to be discrete. At each time step, each vertex i contacts *at most one* of its neighbors j with probability P_{ij}. If either i or j has the information at $t-1$, then both vertices have it at time t.

For the single-piece dissemination algorithm, the ε-dissemination time is defined as

$$T_1(\varepsilon) = \sup_{\upsilon \in V} \inf\{t : \Pr(S(t) \neq V | S(0) = \upsilon) \leq \varepsilon\}. \tag{4}$$

The right hand side of this definition accounts for the maximal time at which the set $S(t)$ is inequivalent to V with probability no greater then ε, assuming that initially the set $S(t = 0)$ consisted of a single vertex υ.

The ε-dissemination time for the single-piece dissemination algorithm may be expressed through the conductance (1) as [2]

$$T_1(\varepsilon) = O\left(\frac{\log n + \log \varepsilon^{-1}}{\Phi(P)}\right). \tag{5}$$

This expression tells us explicitly how the ε-dissemination time depends on the structure of underlying network, i.e. on its conductance. For the complete graph the ε-dissemination time is given by $T_1^c(\varepsilon) = O(\log n)$, which is the upper bound for single-piece dissemination algorithm performance in any network. For the ring graph the ε-dissemination time is exponentially larger comparing to the previous case, i.e. $T_1^r(\varepsilon) = O(n \log n)$. It is important to note that information dissemination on a ring can be performed as fast as $O(n)$ by setting a simple intuitive rule, for example, 'always send information to the left neighbor'. But, gossip algorithms have no account for network structure, which is the key for their universality. Moreover, the gossip algorithms on a ring are just logarithmically slower then the intuitive strategy, which is practical.

2.2. Multi-piece dissemination

In contrast to single-piece dissemination algorithm, where just a single vertex has the information initially, in multi-piece dissemination each vertex wants to spread its own information to all the other vertices as quickly as possible. Let $M = \{m_1, ..., m_n\}$ denote the set of messages at time $t = 0$. As before each vertex contacts at most one of its neighbors at each time step. During the contact, the vertices exchange all information they don't have. The ε-dissemination time is defined as

$$T_M(\varepsilon) = \inf\{t : \Pr\left(\bigcup_{i=1}^{n} S_i(t) \neq M | S_i(0) = m_i\right) \leq \varepsilon\}, \tag{6}$$

i.e. the maximal time at which the information at each vertex is inequivalent to the initial set M with probability no greater then ε. The ε-dissemination time is expressed through the mean conductance (3) as [2]

$$T_M(\varepsilon) = O\left(\frac{\hat{\Phi}(P) \log \varepsilon^{-1}}{n}\right). \tag{7}$$

For the complete graph the ε-dissemination time is given by $T_M^c(\varepsilon) = O(n \log^2 n)$, which is the upper bound for multi-piece dissemination algorithm performance in any network. For the ring graph, in contrast, the ε-dissemination time is exponentially smaller, i.e. $T_M^r(\varepsilon) = O(n^2 \log n)$.

3. Gossip algorithms in quantum networks

Eqs. (5) and (7) unambiguously define performance of gossip algorithms through conductance (1) and it's mean (3) for any clas-