Physics Letters A 380 (2016) 3970–3976

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

A minimalist approach to conceptualization of time in quantum theory

Hitoshi Kitada [a], Jasmina Jeknić-Dugić [b], Momir Arsenijević [c], Miroljub Dugić [c,*]

[a] Department of Mathematics, Metasciences Academy, Japan
[b] Department of Physics, Faculty of Science and Mathematics, University of Niš, Serbia
[c] Department of Physics, Faculty of Science, University of Kragujevac, Serbia

ARTICLE INFO

Article history:
Received 20 July 2016
Received in revised form 7 October 2016
Accepted 10 October 2016
Available online 14 October 2016
Communicated by P.R. Holland

Keywords:
Foundations of quantum mechanics
Functional analytical methods
Lagrangian and Hamiltonian approach

ABSTRACT

Ever since Schrödinger, Time in quantum theory is postulated Newtonian for every reference frame. With the help of certain known mathematical results, we show that the concept of the so-called Local Time allows avoiding the postulate. In effect, time appears as neither fundamental nor universal on the quantum-mechanical level while being consistently attributable to every, at least approximately, closed quantum system as well as to every of its (conservative or not) subsystems.

1. Introduction

Schrödinger's Quantum Mechanics in [1–3], is timeless when he introduced his fundamental equation as a time-independent equation

$$H\psi = E\psi. \tag{1}$$

Here $E \in \mathbb{R}$ and the Hamiltonian H is of the form

$$H = \frac{\hbar^2}{2m}p^2 + V(x), \quad V(x) = -\frac{e^2}{|x|}, \tag{2}$$

where

$$p = \frac{1}{i}\frac{\partial}{\partial x} = \frac{1}{i}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) \tag{3}$$

is the momentum operator conjugate to the position operator $x = (x_1, x_2, x_3)$. With this stationary Schrödinger equation, he could successfully give an explanation of the spectral structure of hydrogen atoms, showing that his formulation of quantum mechanics as the eigenvalue problem of a partial differential operator is valid. Later he proved in [4] that his formulation is equivalent with Heisenberg's formulation of QM. Without loss of generality, we assume $m = 1$ later on.

In the subsequent part [5] he emphasized the necessity to give a time-dependent expression of the equation in order to treat the nonconservative systems, and gave a time-dependent equation for general Hamiltonians

$$\frac{\hbar}{i}\frac{d\psi}{dt}(t) + H\psi(t) = 0. \tag{4}$$

Schrödinger then applied the equation to some time-dependent perturbations with an emphasis of the advantage of the time-dependent approach. He however gave no justification for the notion of time which is assumed for the equation. That is, "time" is postulated [5] to be unique and universally valid throughout the universe as Newton put it in his *Principia Mathematica*.

Exactly the same physical nature of time is assumed for the standard text-book approach to quantum dynamics that is based on the unitary operator $U(t)$, which defines a dynamical map for quantum systems, $\Psi(t) = U(t)\Psi(t = 0)$. Hence we can detect the following two assumptions (postulates) built in the fundamental equation for quantum systems dynamics. The first assumption is the equation's mathematical form provided by eq. (4), which here we adopt without modification. The second assumption is that quantum dynamics unfolds within the classical Newtonian universal (global) time. However, at least as a logical possibility, removing the second assumption is not excluded and, if successful, might make the quantum foundations even more efficient—the less number of postulates, the better theory.

Avoiding this assumption is not a trivial task, which we undertake in this paper. Rejecting the in-advance-agreed role of "physical time" for the parameter t in the unitary operator $U(t)$ elevates to the following two related problems. First, if not in advance, then certainly *a posteriori* the role of the parameter t as physical time should be rigorously established; non-rigorous procedures typically

* Corresponding author.
E-mail address: dugic@open.telekom.rs (M. Dugić).

http://dx.doi.org/10.1016/j.physleta.2016.10.010
0375-9601/

assume certain additional rules and assumptions, often of the interpretational relevance, that here we are not interested in. Second, without a postulate or an interpretational framework, it is not obvious how to link the time-independent Hamiltonian of closed system with the notion of time. These subtle points are regarded in Sections 4 and 5 with the general mathematical basis provided in Section 2. As a result, in Sections 3 and 4 we emphasize a possibility to introduce a notion of time for an arbitrary (including many-particle) *closed* system with the *time-independent* Hamiltonian. We perform *without* resorting to any *ad hoc* procedures or additional assumptions—such as existence of the system's environment, be it classical [6] or not, or time quantization [7], or in-advance-agreed character of physical time. Expectably, such possibility comes at certain price, which in our approach is that time is neither fundamental nor universal on the quantum-mechanical level, and can be recognized as the so-called (quantum-mechanical) local time [8].

2. *N*-particle system

In this section we consider a general *conservative* (i.e. *closed*) quantum mechanical system consisting of N particles; we take a unit system such that $\hbar = 1$. For such a system of $N(\geq 2)$ quantum mechanical particles with mass $m_i(>0)$ located at $r_i \in \mathbb{R}^3$ $(i = 1, \ldots, N)$, Hamiltonian [see also (38) in Appendix A]

$$\begin{aligned} H &= -\sum_{i=1}^{N} \frac{1}{2m_i}\Delta_{r_i} + V(x) = \sum_{i=1}^{N} \frac{1}{2m_i}\left(\frac{1}{i}\frac{\partial}{\partial r_i}\right)^2 + V(x), \\ V(x) &= \sum_{1\leq i<j\leq N} V_{ij}(x_{ij}), \end{aligned} \tag{5}$$

where $\Delta = \Delta_{r_i} = \left(\frac{\partial}{\partial r_i}\right)^2 = \sum_{j=1}^{3} \frac{\partial^2}{\partial r_{ij}^2}$ $(r_i = (r_{i1}, r_{i2}, r_{i3}) \in \mathbb{R}^3)$ is Laplacian and $V_{ij}(x_{ij})$ $(x_{ij} = r_i - r_j)$ is a pair potential working between the pair of particles i and j. When we consider the relative motion of N particles, we can separate the motion of the center of mass as follows. The center of mass of this N-particle system is

$$X_C = \frac{m_1 r_1 + \cdots + m_N r_N}{m_1 + \cdots + m_N}. \tag{6}$$

Defining the Jacobi coordinates by

$$\begin{aligned} &x_i = (x_{i1}, x_{i2}, x_{i3}) = r_{i+1} - \frac{m_1 r_1 + \cdots + m_i r_i}{m_1 + \cdots + m_i} (\in \mathbb{R}^3), \\ &(i = 1, \ldots, N-1) \end{aligned} \tag{7}$$

and corresponding conjugate momentum operators by

$$P_C = \frac{1}{i}\frac{\partial}{\partial X_C}, \quad p_i = \frac{1}{i}\frac{\partial}{\partial x_i} = \frac{1}{i}\left(\frac{\partial}{\partial x_{i1}}, \frac{\partial}{\partial x_{i2}}, \frac{\partial}{\partial x_{i3}}\right), \tag{8}$$

we decompose the Hilbert space $L^2(\mathbb{R}^{3N})$ as a tensor product $L^2(\mathbb{R}^{3N}) = L^2(\mathbb{R}^3) \otimes \mathcal{H}$, $\mathcal{H} = L^2(\mathbb{R}^{3n})$ with $n = N-1$. Accordingly the Hamiltonian H in (5) is decomposed as follows.

$$\begin{aligned} &H = H_C \otimes I + I \otimes \tilde{H}, \\ &\tilde{H} = \tilde{H}_0 + V, \quad H_C = \frac{1}{\sum_{j=1}^{N} m_j} P_C^2, \\ &\tilde{H}_0 = \sum_{i=1}^{N-1} \frac{1}{2\mu_i} p_i^2, \\ &\mu_i^{-1} = m_{i+1}^{-1} + (m_1 + \cdots + m_i)^{-1} \quad (i = 1, \ldots, n). \end{aligned} \tag{9}$$

Here I denotes the identity operator. For real potentials $V_{ij}(x_{ij})$, H in (5) and $\tilde{H}$ in (9) define self-adjoint operators in the Hilbert spaces $L^2(\mathbb{R}^{3N})$ and $\mathcal{H} = L^2(\mathbb{R}^{3n})$, respectively, and the relative motion of the N-particles is described by the Hamiltonian $\tilde{H}$ in $\mathcal{H} = L^2(\mathbb{R}^{3n})$.

By (9), $H_C \otimes I$ is a nonnegative selfadjoint operator in $L^2(\mathbb{R}^3)$ and describes the free motion of the center of mass of the N-particle system whose property is well-known. Our main concern is thus about the relative motion of the N particles. Henceforth we will write

$$H = \tilde{H}, \quad H_0 = \tilde{H}_0, \tag{10}$$

and consider the Hamiltonian in $\mathcal{H} = L^2(\mathbb{R}^{3n})$

$$H = H_0 + V = \sum_{i=1}^{N-1} \frac{1}{2\mu_i} p_i^2 + V(x). \tag{11}$$

We note that H is defined solely through the configuration operators $x = (x_1, \ldots, x_{N-1})$ and conjugate momentum operators $p = (p_1, \ldots, p_{N-1})$. Thus time-independent QM is completely determined through position and momentum operators (x, p), since the corresponding stationary time-independent Schrödinger equation (1) is written as follows.

$$(H - \lambda I)\psi = 0. \tag{12}$$

This equation has non-zero solution $\psi \in \mathcal{H}$ only when λ is an eigenvalue of H: $\lambda \in \sigma_p(H)$. A complex number λ is said to belong to the resolvent set $\rho(H)$, when (12) has only a trivial solution $f = 0$ and the bounded inverse $(H - \lambda I)^{-1} : \mathcal{H} \to \mathcal{H}$ exists. $R(\lambda) = R_H(\lambda) = (H - \lambda I)^{-1}$ is called the resolvent at $\lambda \in \rho(H)$ of H. We review some concepts on spectrum $\sigma(H)$ of a selfadjoint operator H.

Definition 1.

1) The set of all complex numbers $\lambda \in \mathbb{C} \setminus \rho(H)$ is called the spectrum of H and denoted by $\sigma(H)$. For a selfadjoint operator H it is trivial to see that $\sigma(H) \subset \mathbb{R}$.
2) We denote the resolution of the identity corresponding to a selfadjoint operator H by $E_H(\lambda)$ $(\lambda \in \mathbb{R})$:

$$\begin{aligned} &E_H(\lambda)E_H(\mu) = E_H(\min(\lambda, \mu)), \\ &\text{s-}\lim_{\lambda\to-\infty} E_H(\lambda) = 0, \quad \text{s-}\lim_{\lambda\to\infty} E_H(\lambda) = I, \\ &E_H(\lambda + 0) = E_H(\lambda), \\ &f(H) = \int_{-\infty}^{\infty} f(\lambda) dE_H(\lambda) \quad (\forall f \in C(\mathbb{R})), \end{aligned} \tag{13}$$

where $E_H(\lambda + 0) = \text{s-}\lim_{\mu\downarrow\lambda} E_H(\mu)$ and $C(\mathbb{R})$ is the set of all complex-valued continuous functions on $\mathbb{R}$. An operator-valued measure $E_H(B)$ $(B \subset \mathbb{R}$: Borel set) is defined by the relation $E_H((a, b]) = E_H(b) - E_H(a)$ for $-\infty < a < b < +\infty$.
3) Set $P(\lambda) = E_H(\lambda) - E_H(\lambda - 0)$ $(\lambda \in \mathbb{R})$. We note that $P(\lambda) \neq 0$ iff λ is an eigenvalue of H. When $\lambda \in \sigma_p(H)$, $P(\lambda)\mathcal{H}$ is the eigenspace of H for $\lambda \in \sigma_p(H)$. The pure point spectral subspace (or eigenspace) $\mathcal{H}_p(H)$ for H is defined as the closed linear hull of the set

$$\bigcup_{\lambda\in\mathbb{R}} P(\lambda)\mathcal{H}. \tag{14}$$

Eigenprojection P_H is the orthogonal projection onto $\mathcal{H}_p(H)$.
4) The continuous spectral subspace for H is defined by

$$\begin{aligned} &\mathcal{H}_c(H) = \{\psi \mid E_H(\lambda)\psi \\ &\text{is continuous with respect to } \lambda \in \mathbb{R}\}, \end{aligned} \tag{15}$$

and the absolutely continuous spectral subspace for H by