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A criterion of orthogonality on the assumption and restrictions in subgrid-scale modelling of turbulence

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ABSTRACT

In order to shed light on understanding the subgrid-scale (SGS) modelling methodology, we analyze and define the concepts of assumption and restriction in the modelling procedure, then show by a generalized derivation that if there are multiple stationary restrictions in a modelling, the corresponding assumption function must satisfy a criterion of orthogonality. Numerical tests using one-dimensional nonlinear advection equation are performed to validate this criterion. This study is expected to inspire future research on generally guiding the SGS modelling methodology.

1. Introduction

The large-eddy simulation (LES) technique has achieved great success in the last half-century in both academic community and industry applications [1–3]. Numerous SGS models have been introduced under various frameworks. The classification and comparison of these models can be found in literature [1,4–8]. These subgrid-scale (SGS) models clearly show the variety of the turbulence community and illustrate the common interest of SGS modelling. However, confusions are also involved in these distinctive SGS models: (i) in the engineering community, researchers are usually not able to *a-priori* suggest the best SGS model for a typical problem and (ii) in the turbulence community, researchers may also be confused on the (necessary) criterion of constructing an SGS model. Reference [9] concluded three basic requirements for evaluating a good SGS model, which is helpful for clarifying the first confusion. However, the second point is still not clear, which indeed calls for better understanding on SGS modelling procedure. Specifically, as researchers on SGS modelling, how should we "correctly" involve physical and mathematical restrictions to model the SGS quantities?

In recent years, along with developing the SGS models using the Kolmogrov equation of filtered velocity (KEF) [10–14], we have also been curious on clarifying the high-level methodology of SGS modelling. In fact, we found that when employing these SGS models, if we calculate the model coefficient dynamically, the LES calculations are usually unstable. Instead, a constant model coefficient usually leads to better performance. This fact has been discussed in Refs. [3,13,15], but we did not manage to give a convincing theoretical explanation. In Ref. [3] we summarized the various attempts of this modelling methodology, and phenomenally guessed that there are some conflicts among the *assumptions* and *restrictions* in SGS modelling.

In order to better explain the research context and to define the terms, here we review our analysis on the procedure of SGS modelling methodology. In general, by reviewing the existing SGS models, we describe the procedure of SGS modelling as the following two steps [3,16,17]:

(i) Any SGS model should be based on a certain *assumption* on the SGS motion, that is, we need to assume a formulation for SGS quantities (in particular, the SGS stress tensor). However, there are always undetermined factors in this *assumption*.
(ii) A complete SGS model should employ a certain closure method to determine the unknown factors mentioned in step (i). This closure implies one or more *restrictions* in either physical or mathematical frameworks.

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Let us comment on the definitions of *assumptions* and *restrictions*, which are abstract conceptions of the above two-step description. There are various *assumptions* such as the eddy-viscosity assumption [18], the scale-similarity assumption [19], the gradient diffusion assumption [20], the velocity increment assumption [13,21], *etc*. We conclude that *an assumption assumes a (local) similarity between GS and SGS quantities*. Usually there can not be two or more *assumptions* at the same time in one SGS model. By contrast, *a restriction is a physical or mathematical simplification which is employed in the SGS modelling closure*. There are various types of *restrictions*, for example the inviscid simplification [10], the scaling laws [15,22–24], the filter similarity [25], the velocity profile restriction [26], *etc*. We remark that there can be multiple *restrictions* in one SGS model, for example in Ref. [10] four *restrictions* are employed: the inviscid simplification, the scaling law, the stationary simplification of structure function, and the subgrid feasibility for Taylor expansion. In general, the reason of choosing multiple *restrictions* is that we hope the generated turbulence can satisfy more physical or mathematical laws. Whereas, sometimes multiple *restrictions* lead to non-negligible error but we do not know the reason.

As introduced above, from previous numerical practices we intuitively feel that there are some conflicts among the *assumptions* and *restrictions* in SGS modelling. Aiming at demonstrating that, considering the complexity of different *restrictions*, in this contribution we choose a simple type of *restriction*, *i.e.*, the stationary *restriction* for a statistical quantity. We will show by a generalized derivation that if there are multiple stationary *restrictions* in a modelling, the corresponding *assumption* function must satisfy a criterion of orthogonality. Numerical tests in one-dimensional nonlinear advection equation are performed to support this criterion.

2. Theoretical analysis

We consider here the general formulation of a nonlinear partial differential equation

$$\frac{\partial u}{\partial t} = f(u), \tag{1}$$

where t is time and u is a continuous physical quantity. For simplicity, here we consider u as a scalar function. $f(u)$ is a nonlinear function, which needs an SGS modelling in numerical applications. Defining $\bar{\bullet}$ the grid-scale filter operator, Eq. (1) can be rewritten as

$$\frac{\partial \bar{u}}{\partial t} = f(\bar{u}) + \tau, \tag{2}$$

with τ the SGS term which needs to be modelled. As introduced in the previous section, we should involve one *assumption* and some *restriction(s)*. In this contribution we only consider typical stationary *restrictions* for a statistical quantity, which can be written as

$$\frac{\partial \langle h_i(\bar{u})\rangle}{\partial t} = 0, \tag{3}$$

with $i = 1, 2, ..., n$ and n the number of *restrictions*, h_i a nonlinear function, and $\langle \bullet \rangle$ the ensemble average. We remark that, for example, this type of *restrictions* can refer to the stationary simplification of resolved longitudinal structure function $D_{ll}^{<}$ [10,11], and the stationary simplification of Lagrangian velocity gradient correlation [27]. Here we consider that there are two *restrictions* h_1 and h_2, and use the following methodology to involve an *assumption* to link the resolved and SGS quantities.

Multiplying both sides of Eq. (2) with $\frac{dh_1(\bar{u})}{d\bar{u}}$ and by noting that $\frac{\partial h_1(\bar{u})}{\partial t} = \frac{dh_1(\bar{u})}{d\bar{u}} \frac{\partial \bar{u}}{\partial t}$, we can obtain

$$\frac{\partial h_1(\bar{u})}{\partial t} = \frac{dh_1(\bar{u})}{d\bar{u}} f(\bar{u}) + \frac{dh_1(\bar{u})}{d\bar{u}} \tau. \tag{4}$$

Taking ensemble average on Eq. (4) and by noting the *restriction* (3), it is

$$\left\langle \frac{dh_1(\bar{u})}{d\bar{u}} f(\bar{u}) + \frac{dh_1(\bar{u})}{d\bar{u}} \tau \right\rangle = 0. \tag{5}$$

In order to satisfy the statistical property (5) by keeping the instantaneous scale-similarity, in SGS modelling we usually introduce an *assumption* here, which involves a function $Z(\bar{u})$ of the resolved quantities $\bar{u}$ with zero average and suppose that it is locally similar to the average-zero SGS expression $\frac{dh_1(\bar{u})}{d\bar{u}} f(\bar{u}) + \frac{dh_1(\bar{u})}{d\bar{u}} \tau$. This is an abstraction of practical assumptions, such as the viscosity assumption which assumes a local similarity between the trace-zero SGS tensor $\tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij}$ (which is analogy to $\frac{dh_1(\bar{u})}{d\bar{u}} f(\bar{u}) + \frac{dh_1(\bar{u})}{d\bar{u}} \tau$) and the resolved strain rate $\bar{S}_{ij}$ (which is analogy to $Z(\bar{u})$) with δ Kronecker delta. We therefore write this local similarity as

$$\frac{dh_1(\bar{u})}{d\bar{u}} f(\bar{u}) + \frac{dh_1(\bar{u})}{d\bar{u}} \tau = Z(\bar{u}), \tag{6}$$

which yields the expression of the SGS quantity

$$\tau = \frac{1}{\frac{dh_1(\bar{u})}{d\bar{u}}} \left(Z(\bar{u}) - \frac{dh_1(\bar{u})}{d\bar{u}} f(\bar{u}) \right). \tag{7}$$

The above procedure describes a generalized methodology for determining the SGS quantity with one *assumption* and one *restriction*. However, as introduced in the previous section, there are usually multiple *restrictions* in an SGS modelling. It is then interesting to see whether other *restrictions* are able be satisfied under this procedure. Substituting Eq. (7) to (2) for $i \neq 1$, and multiplying both sides with $\frac{dh_i(\bar{u})}{d\bar{u}}$, we obtain

$$\frac{\partial h_i(\bar{u})}{\partial t} = Z(\bar{u}) \left. \frac{dh_i(\bar{u})}{d\bar{u}} \middle/ \frac{dh_1(\bar{u})}{d\bar{u}} \right. . \tag{8}$$

Taking ensemble average, it is shown that the *restrictions* (3) lead to

$$\left\langle Z(\bar{u}) h_i'(\bar{u}) \middle/ h_1'(\bar{u}) \right\rangle = 0. \tag{9}$$

If we consider $\bar{u}$ as a random variable with probability density function (PDF) $\rho(\bar{u})$, Eq. (9) can be rewritten as

$$\int_{-\infty}^{+\infty} Z(s) H_i(s) \rho(s) ds = 0, \tag{10}$$

where H_i is defined as $H_i = h_i' / h_1'$, and s is the variable of integration in probability space for replacing $\bar{u}$ in Eq. (9). This can also be further rewritten as a formula of inner product

$$Z \cdot H_i := \int_{-\infty}^{+\infty} Z(s) H_i(s) \rho(s) ds = 0. \tag{11}$$

This inner product then defines an inner product space with a weight function ρ. The *restrictions* h_i correspond to a series of elements H_i in this space, while $Z(s)$ must be orthogonal to the linear subspace $\mathrm{Vect}(H_1, H_2, ..., H_n)$. This describes a criterion of orthogonality on the *assumption* and *restrictions*.

3. Numerical tests

In this section we select the most simple nonlinear partial differential equation, *i.e.*, the one-dimensional nonlinear advection equation, or say, the one-dimensional inviscid Burgers equation, to validate the conclusions in the present contribution. We remark that the analysis of the previous section might also be used

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