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On reaction processes with a logarithmic-diffusion

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ABSTRACT

We study formation of patterns in reaction processes with a logarithmic-diffusion: $u_t = (\ln u)_{xx} + R(u)$. For the generic $R = u(1-u)$ case the problem of travelling waves, TW, is mapped into a linear one with the propagation speed λ selected by a boundary condition, b.c. at the far away upstream. Dirichlet b.c. relaxes the process *into a steady state*, whereas convective b.c. $u_x + hu = 0$, leads the system into a heating (cooling) TW for $h < 1$ $(1 < h)$ or, if $h = 1$, into an equilibrium. We derive explicit solutions of symmetrically expanding waves and of formations which collapse in a finite time. Both are shown to be attractors of classes of initial excitations. For a bi-stable reaction $R = -u(\alpha - u)(1-u)$ we show that for $\alpha < 1/3$ the system may evolve into a TW, an equilibrium, an expanding formation or to collapse. The $1/3 < \alpha$ regime admits either a cooling TW or a collapse. Few other transport processes are outlined in the appendix.

1. Introduction

In the last two decades logarithmic diffusion gained remarkable notoriety due to a variety of applications in both physics and mathematics, see refs. [1,2] and references therein. An additional, not least important, argument in favor of further exploration of transport with a logarithmic diffusion is its remarkable level of solvability, a rare feature in nonlinear processes, which reveals rich and challenging phenomena. In the present work we shall unfold some of the properties due to coupling of logarithmic diffusion with reaction. In the next two sections we present the quadratic, Fisher-KPP, reaction whereas in sect. 4 we address the bistable case. In the appendix we extend the presented method to other reaction–diffusion processes. In particular we discuss formation of cavity due to radiation.

2. The Fisher-KPP reaction

We start with the classical Fisher-KPP reaction

$$u_t = (\ln u)_{xx} + u(1-u), \quad x \in \mathcal{R} \tag{1}$$

with

$$u(-\infty) = 1 \;\; \textit{and} \;\; u(+\infty) = 0. \tag{2}$$

Unlike the standard Fisher-KPP case wherein precursor's dynamics is linear, *the presented process is essentially nonlinear down to the ground state*. Moreover, if the wave's precursor decays as $u \sim e^{-ax}$ then, unlike the classical case wherein the flux $-u_x \sim e^{-ax}$, here flux $-u_x/u \sim a$ and thus remains finite as $x \to \infty$. This has a fundamental impact on the overall dynamics and may cause a complete extinction of the process within a finite time [1].

With the classical KPP results in mind we turn to find Travelling Waves, TW, of (1)–(2). Let $z = x - 2\lambda t$, then

$$2\lambda \frac{du}{dz} + \frac{d^2 \ln u}{dz^2} + u(1-u) = 0. \tag{3}$$

To solve Eq. (3) we introduce a map

$$d\zeta = u dz \quad \textit{or} \;\; \zeta = \int_{-\infty}^{z} u(\eta) d\eta, \tag{4}$$

under which Eq. (3) becomes linear

$$u\left[2\lambda u' + u'' - u + 1\right] = 0, \quad \textit{with} \;\; a' = \frac{da}{d\zeta}, \tag{5}$$

and the b.c.: $u(\zeta = -\infty) = 1$ and $u(\zeta = \zeta_0) = 0$. Its solution, we may assume $\zeta_0 = 0$, is

$$u(\zeta) = \left[1 - e^{\gamma \zeta}\right]_+, \quad \textit{where} \;\; \lambda = \frac{1-\gamma^2}{2\gamma}, \tag{6}$$

or, since $0 < \gamma$ for the solution to stay bounded, $\gamma^{-1} = \lambda + \sqrt{1+\lambda^2}$. In z coordinates we have

$$u(z) = \frac{1}{1+e^{\gamma z}}. \tag{7}$$

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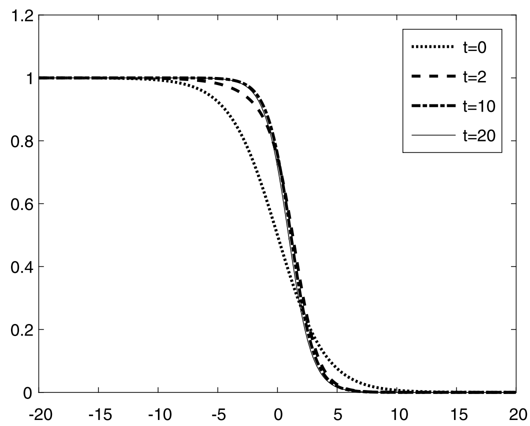

Fig. 1. The initial kink $u(x,0) = 1/(1+e^{x/2})$ converges to the steady state of Eq. (1) with $h = 1$. Note that everywhere but in Fig. 8 convective b.c. were used.

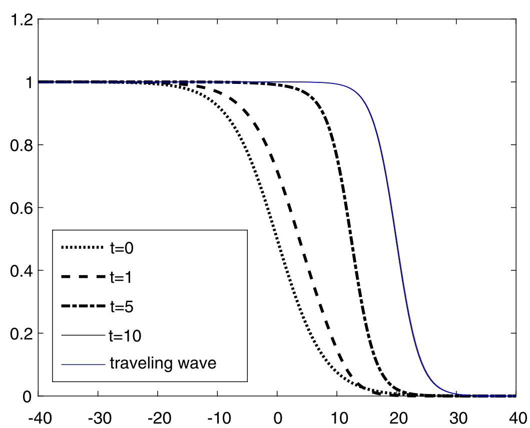

Fig. 2. The initial kink $u(x,0) = 1/(1+e^{x/4})$ converges to a heating TW with $\lambda = 1/2$ and $h = 1/2$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

Speed selection. So far the speed λ is arbitrary and may take both *positive and negative values*, with the latter corresponding to a cooling wave receding to the left. The later being a novel feature of the logarithmic diffusion. However, with the flux being finite at $+\infty$, to assure uniqueness of the solution one has to be more specific about the action there, which is to say that one has to append the problem with an additional boundary condition upstream. The specific form of the flux in the present problem makes it natural to impose

$$u_x + hu = 0 \quad at\ x = \infty, \tag{8}$$

which uniquely relates the convection coefficient h with the propagation speed

$$\lambda = \frac{1-h^2}{2h}. \tag{9}$$

In particular, the choice $h = 1$ begets a steady state

$$u(x) = \frac{1}{1+e^{x+x_*}}, \quad x_* = const., \tag{10}$$

which means that when $h = 1$ heat production within the domain is balanced exactly by heat removed at infinity. Fig. 1 describes a typical convergence of an initial kink-like excitation into an equilibrium. Taking $h < 1$ reduces the amount of heat which leaves at infinity. The excess of the heat generated in the domain induces a heating wave propagating to the right and thus $\lambda > 0$ (see Fig. 2 where the chosen $h = 1/2$ induces a heat wave with $\lambda = 3/4$). When $1 < h$ more heat leaves the domain than generated within.

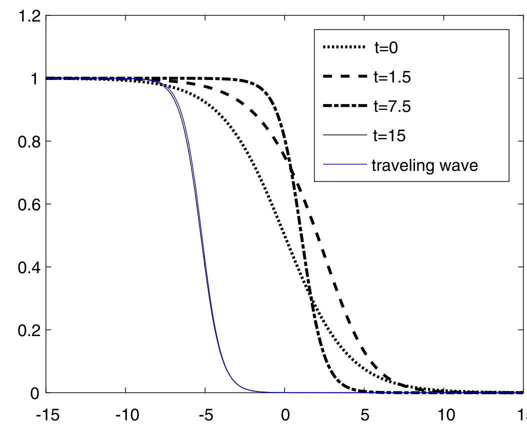

Fig. 3. The initial kink $u(x,0) = 1/(1+e^{x/2})$ converges to a receding cooling TW of Eq. (1) with $\lambda = -1/2$ and $h = (1+\sqrt{5})/2$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

This induces a cooling wave receding to the left, i.e., $\lambda < 0$. Fig. 3 displays such scenario.

Alternatively, one may prescribe u at a finite distance L: $u(L) = U_0 < 1$. Now only stationary states (10) are admissible with x_* related to U_0 and L via

$$x_* = \ln(\frac{1}{U_0} - 1) - L. \tag{11}$$

Note the flux at $x = L$

$$-\frac{u_x}{u}|_{x=L} = \frac{1}{1+e^{-L}} \tag{12}$$

and it attains its limiting value 1 for $L >> 1$. Very much as in Fig. 1, the stationary state with Dirichlet boundary condition imposed at L is an attractor: initial profiles located above (below) the equilibrium propagate to the right (left) before settling into a steady-state (not shown).

3. Expanding waves

We now explore a wider family of solutions which also provides an alternative path to the TW solutions (for an application to other transport equations see the Appendix). Let $v = 1/u$, then in terms of v Eq. (1) reads

$$v_t = v v_{xx} - v_x^2 + \omega^2(1-v), \quad x \in \mathcal{R}, \tag{13}$$

where ω was added for a better trace of reaction's impact. We seek solution in the form

$$v(x,t) = A(t) + B(t) f(x) \tag{14}$$

which satisfies

$$\dot{A} - \omega^2(1-A) = -(\dot{B} + \omega^2 B) f + ABf'' + B^2[Q(f)] \tag{15}$$

and $Q(f) \doteq ff'' - f'^2$. We constrain $Q(f)$ to satisfy

$$ff'' - f'^2 = \alpha_0 + \alpha_1 f, \quad \alpha_0, \alpha_1\ consts. \tag{16}$$

Further developments depend on α_0 and α_1.

1) Let $\alpha_0 = \alpha_1 = 0$, then $f(x) = \exp(\gamma x)$. Solving for A and B we find for both γ and the solution an $\omega \neq 1$ extension of Eqs. (6) and (7).

2) Let $\alpha_0 = 0$ and $\alpha_1 = -2$. Then $f(x) = x^2$ and thus

$$u(x,t) = \frac{1}{A(t) + B(t)x^2}, \tag{17}$$

with $A(t)$ found solving