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Parametric instabilities in shallow water magnetohydrodynamics of astrophysical plasma in external magnetic field

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ABSTRACT

This article deals with magnetohydrodynamic (MHD) flows of a thin rotating layer of astrophysical plasma in external magnetic field. We use the shallow water approximation to describe thin rotating plasma layer with a free surface in a vertical external magnetic field. The MHD shallow water equations with external vertical magnetic field are revised by supplementing them with the equations that are consequences of the magnetic field divergence-free conditions and reveal the existence of third component of the magnetic field in such approximation providing its relation with the horizontal magnetic field. It is shown that the presence of a vertical magnetic field significantly changes the dynamics of the wave processes in astrophysical plasma compared to the neutral fluid and plasma layer in a toroidal magnetic field. The equations for the nonlinear wave packets interactions are derived using the asymptotic multiscale method. The equations for three magneto-Poincare waves interactions, for three magnetostrophic waves interactions, for the interactions of two magneto-Poincare waves and for one magnetostrophic wave and two magnetostrophic wave and one magneto-Poincare wave interactions are obtained. The existence of parametric decay and parametric amplifications is predicted. We found following four types of parametric decay instabilities: magneto-Poincare wave decays into two magneto-Poincare waves, magnetostrophic wave decays into two magnetostrophic waves, magneto-Poincare wave decays into one magneto-Poincare wave and one magnetostrophic wave, magnetostrophic wave decays into one magnetostrophic wave and one magneto-Poincare wave. Following mechanisms of parametric amplifications are found: parametric amplification of magneto-Poincare waves, parametric amplification of magnetostrophic waves, magneto-Poincare wave amplification in magnetostrophic wave presence and magnetostrophic wave amplification in magneto-Poincare wave presence. The instabilities growth rates are obtained respectively.

1. Introduction

Plasma in various stars and planets is described by magnetohydrodynamics of thin fluid layer with a free surface in the gravity field. As an example we refer to solar tachocline flows (a thin layer inside the sun located above the convection zone) [11,20,21], neutron stars atmosphere dynamics [9,22], accreting matter flows in the neutron stars [12], tidally synchronized exoplanets with magnetoactive atmospheres [8,10,2]. MHD shallow water approximation [6] and quasigeostrophic MHD approximation [6,18,1] are used to describe such flows of astrophysical plasma. The present article is concerned with the study of weakly non-linear waves interactions in shallow water magnetohydrodynamics. We consider MHD flows in plasma layer for which the characteristic horizontal scales significantly exceed the scales of vertical variations. Therefore we assume the layer to be thin and use the shallow water approximation. The MHD equations in shallow water approximation are the alternative for heavy fluid MHD equations in the case where the layer of incompressible inviscid fluid in gravity field with a free surface is considered. The non-inertial reference frame is rotating with the fluid layer. The MHD shallow water equations are obtained by depth-averaging from general incompressible MHD equations. The pressure is assumed to be hydrostatic and layer height is assumed to be much smaller than characteristic horizontal linear scale of plasma layer [6,7,13–15,3,4,23]. The obtained equations play the important role in astrophysical plasmas studies as do the classical shallow water equations in neutral fluid hydrodynamics. It should be noted that in hydrodynamics of neutral fluid in shallow water approximation there are only gravitational Poincare waves and the dispersion relation in shallow water equa-

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tions for neutral fluid exclude Poincare waves interactions [19], because in this case phase matching conditions are not fulfilled. In common shallow water magnetohydrodynamics used in solar tachocline studies toroidal magnetic field exists and it leads to existence of two linear modes: fast magnetogravity waves (similar to Poincare waves in neutral fluid) called magneto-Poincare and slow Alfven waves. Simultaneously the dispersion relation doesn't provide phase matching conditions in weakly non-linear interactions [20,6,7]. In present study we consider MHD shallow water equations in the external vertical magnetic field. Such configuration of magnetic field is typical for the neutron stars [9,2] and for the exoplanets [2]. The external vertical magnetic field significantly modifies MHD shallow water equations and in linear approximation two new fast waves appear: magneto-Poincare mode and magnetostrophic mode [9]. We revised the MHD shallow water equations with external vertical magnetic field supplementing them with the equations that are consequences of the magnetic field divergence-free conditions that are satisfied identically. The new supplementary equations reveal the existence of three components of the magnetic field and provide the expressions for variation of the vertical magnetic field. This is not the case in commonly used shallow water equations without external magnetic fields when magnetic field is inevitably horizontal. In the present work we generalize the linear theory of MHD shallow water flows developed in [9] to the case of finite amplitude waves in weakly non-linear approximation. It is shown that the dispersion relations of linear waves in external vertical magnetic field provide the phase matching conditions needed for non-linear interactions. The interactions of wave packets are investigated in shallow water magnetohydrodynamics of rotating plasma layer. The analysis of the dispersion relations for both modes shows that there are several types of three waves interactions: three magneto-Poincare waves, three magnetostrophic waves and also intermode interactions: two magneto-Poincare waves and magnetostrophic wave, two magnetostrophic waves and magneto-Poincare wave. We use multiscale asymptotic method to describe non-linear interactions [16]. The non-linear equations of wave amplitudes interaction are derived for each case. The analysis of obtained non-linear equations for the three-waves interactions shows that there are two types of instabilities: parametric decay and parametric amplification. We found following four types of parametric decay instabilities: magneto-Poincare wave decays into two magneto-Poincare waves, magnetostrophic wave decays into two magnetostrophic waves, magneto-Poincare wave decays into one magneto-Poincare wave and one magnetostrophic wave, magnetostrophic wave decays into one magnetostrophic wave and one magneto-Poincare wave. The instability growth rates are found. Also following four types of parametric amplification mechanisms were investigated: parametric amplification of magneto-Poincare waves, parametric amplification of magnetostrophic waves, magneto-Poincare wave amplification in magnetostrophic wave presence and magnetostrophic wave amplification in magneto-Poincare wave presence. Increments are found for each type of instability.

In section 2 we provide the shallow water MHD equations in rotating frame on a flat surface. Linear solutions and dispersion curves are provided. The analysis of phase matching conditions is shown the possibility of three-waves interactions. In section 3 we derive the equations for the slowly varying amplitudes of three-waves interactions in shallow water magnetohydrodynamics in the external vertical magnetic field. In section 4 the obtained equations of three-waves interactions are used to analyse physical effects of weakly non-linear interactions of magneto-Poincare and magnetostrophic waves. Parametric decays and parametric amplifications are analysed. The obtained results are formulated in section 5. We revise the shallow water MHD equations in external magnetic field in the appendix.

2. Shallow water approximation. Qualitative analysis of MHD flows of astrophysical plasma

Here, we introduce MHD shallow water equations on a flat plane that describes the flows of thin plasma layer with a free surface in a gravity field in the presence of rotation. Linear solutions of this system are used to qualitatively analyse the dispersion curves for linear waves and to determine the phase matching conditions that make the interwave interaction possible.

The MHD shallow water equations are obtained from the full set of three-dimensional MHD equations. These shallow water equations are obtained for the thin fluid layer with a free surface in the gravitational field which is rotating with the Coriolis parameter f in the external vertical magnetic field B_0 [9]. For magnetic field normalized by the factor $(4\pi\rho)^{-\frac{1}{2}}$, in which ρ is the (assumed constant) density the equations take the form:

$$\frac{\partial h}{\partial t}+\frac{\partial h v_x}{\partial x}+\frac{\partial h v_y}{\partial y}=0 \tag{2.1}$$

$$\frac{\partial(hv_x)}{\partial t}+\frac{\partial(h(v_x^2-B_x^2))}{\partial x}+gh\frac{\partial h}{\partial x}+\frac{\partial(h(v_xv_y-B_xB_y))}{\partial y} + B_0B_x = fhv_y \tag{2.2}$$

$$\frac{\partial(hv_y)}{\partial t}+\frac{\partial(h(v_xv_y-B_xB_y))}{\partial x}+\frac{\partial(h(v_y^2-B_y^2))}{\partial y}+gh\frac{\partial h}{\partial y} + B_0B_y = -fhv_x \tag{2.3}$$

$$\frac{\partial(hB_x)}{\partial t}+\frac{\partial(h(B_xv_y-B_yv_x))}{\partial y}+B_0v_x=0 \tag{2.4}$$

$$\frac{\partial(hB_y)}{\partial t}+\frac{\partial(h(B_yv_x-B_xv_y))}{\partial x}+B_0v_y=0 \tag{2.5}$$

In (2.1)–(2.5) h is free surface vertical coordinate, v_x, v_y are horizontal velocities in shallow water approximation in the xy-plane, B_x, B_y are horizontal components of depth-averaging magnetic fields in shallow water approximation in the x and y directions respectively, B_0 is an external magnetic field which is directed perpendicular to the xy-plane, f is the Coriolis parameter of the flow. The set (2.1)–(2.5) is the result of integrating the three-dimensional MHD equations over z-axis. We assume the complete pressure (hydrodynamic and magnetic) to be hydrostatic [14,13,15,23]. The first equation of the set (2.1)–(2.5) is the result of integrating the continuity equation, the second and the third ones are the equations of conservation of momentum, the fourth and the fifth ones are the equations of magnetic field variations. This closed set of equations, derived in [9], is complete for analysing linear waves and non-linear interactions. In the limit when $B_0 = 0$ these equations reduce to the common MHD shallow water equations [6]. Indeed, we supplement (2.1)–(2.5) with the equations, that distinguish significantly the MHD shallow water equations with external vertical magnetic field from the equations without vertical magnetic field:

$$\frac{\partial B_z}{\partial t}+B_0\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\right)=0 \tag{2.6}$$

$$\frac{\partial hB_x}{\partial x}+\frac{\partial hB_y}{\partial y}+B_z=0 \tag{2.7}$$

In the traditional derivation of the MHD shallow water equations from the full set of three-dimensional MHD equations in [14,13,23] the vertical component of magnetic field is assumed to be zero. Note that the presence of a vertical magnetic field leads to essential changes of horizontal magnetic field dynamics in shallow water approximation. It should be noted that the horizontal magnetic field is solenoidal in the case without external