Physics Letters A 381 (2017) 1081–1086

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Approximate analytic solutions to coupled nonlinear Dirac equations

Avinash Khare [a], Fred Cooper [b,c], Avadh Saxena [c,*]

[a] Physics Department, Savitribai Phule Pune University, Pune 411007, India
[b] Santa Fe Institute, Santa Fe, NM 87501, USA
[c] Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, NM 87545, USA

ARTICLE INFO

Article history:
Received 22 March 2016
Received in revised form 7 December 2016
Accepted 10 January 2017
Available online 30 January 2017
Communicated by A.P. Fordy

Keywords:
Nonlinear Dirac equation
Scalar and vector interactions
Conservation laws
Solitons

ABSTRACT

We consider the coupled nonlinear Dirac equations (NLDEs) in $1+1$ dimensions with scalar–scalar self-interactions $\frac{g_1^2}{2}(\bar\psi\psi)^2 + \frac{g_2^2}{2}(\bar\phi\phi)^2 + g_3^2(\bar\psi\psi)(\bar\phi\phi)$ as well as vector–vector interactions of the form $\frac{g_1^2}{2}(\bar\psi\gamma_\mu\psi)(\bar\psi\gamma^\mu\psi) + \frac{g_2^2}{2}(\bar\phi\gamma_\mu\phi)(\bar\phi\gamma^\mu\phi) + g_3^2(\bar\psi\gamma_\mu\psi)(\bar\phi\gamma^\mu\phi)$. Writing the two components of the assumed rest frame solution of the coupled NLDE equations in the form $\psi = e^{-i\omega_1 t}\{R_1\cos\theta, R_1\sin\theta\}$, $\phi = e^{-i\omega_2 t}\{R_2\cos\eta, R_2\sin\eta\}$, and assuming that $\theta(x), \eta(x)$ have the *same* functional form they had when $g_3 = 0$, which is an approximation consistent with the conservation laws, we then find approximate analytic solutions for $R_i(x)$ which are valid for small values of g_3^2/g_2^2 and g_3^2/g_1^2. In the nonrelativistic limit we show that both of these coupled models go over to the same coupled nonlinear Schrödinger equation for which we obtain two exact pulse solutions vanishing at $x \to \pm\infty$.

1. Introduction

The nonlinear Dirac (NLD) equation in $1+1$ dimensions [1] has a long history and has emerged as a useful model in many physical systems such as extended particles [2–4], the gap solitons in nonlinear optics [5], light solitons in waveguide arrays and experimental realization of an optical analog for relativistic quantum mechanics [6–8], Bose–Einstein condensates in honeycomb optical lattices [9], phenomenological models of quantum chromodynamics [10], as well as matter influencing the evolution of the universe in cosmology [11]. Further, the multi-component BEC order parameter has an exact spinor structure and serves as the bosonic analog to the relativistic electrons in graphene. To maintain the Lorentz invariance of the NLD equation, the self-interaction Lagrangian is built using the bilinear covariants. Of special interest are scalar bilinear covariant and vector bilinear covariant which have particularly attracted a lot of attention.

Classical solutions of nonlinear field equations have a long history as a model of extended particles [12]. In 1970, Soler proposed that the self-interacting 4-Fermi theory was an interesting model for extended fermions. Later, Strauss and Vasquez [13] were able to study the stability of this model under dilatation and found the domain of stability for the Soler solutions. Solitary waves in the $1+1$ dimensional nonlinear Dirac equation have been studied [14,15] in the past in case the nonlinearity parameter $k = 1$, i.e. the massive Gross–Neveu [16] (with $N = 1$, i.e. just one localized fermion) and the massive Thirring [17] models. In those studies it was found that these equations have solitary wave solutions for both scalar–scalar (S–S) and vector–vector (V–V) interactions. The interaction between solitary waves of different initial charge was studied in detail for the S–S case in the work of Alvarez and Carreras [18] by Lorentz boosting the static solutions and allowing them to scatter.

In a previous paper [19] we extended the work of these preceding authors to the case where the nonlinearity was taken to an arbitrary power κ for both the scalar–scalar and vector–vector couplings and were able to find solitary wave solutions for an arbitrary nonlinearity parameter κ. In this paper we will extend the previous models in a new direction by looking for solitary wave solutions to the problem of two coupled NLDEs and considering the scalar–scalar coupling as well as the vector–vector coupling between the two fields. That is, we assume the interaction Lagrangian has the form $\frac{g_1^2}{2}(\bar\psi\psi)^2 + \frac{g_2^2}{2}(\bar\phi\phi)^2 + g_3^2(\bar\psi\psi)(\bar\phi\phi)$ for the scalar–scalar interaction and for the vector–vector interaction of the form $\frac{g_1^2}{2}(\bar\psi\gamma_\mu\psi)(\bar\psi\gamma^\mu\psi) + \frac{g_2^2}{2}(\bar\phi\gamma_\mu\phi)(\bar\phi\gamma^\mu\phi) + g_3^2(\bar\psi\gamma_\mu\psi)(\bar\phi\gamma^\mu\phi)$.

Here we concern ourselves solely with the question of the existence of approximate analytic solutions to these coupled NLDEs in $1+1$ dimension which reduce to the usual *exact* soliton solutions of the NLDE when the coupling between fields g_3 is set to zero. Of course, there is a huge related literature on solving the related *3 + 1* dimensional two-body Dirac equations that are used

* Corresponding author.
E-mail addresses: khare@physics.unipune.ac.in (A. Khare), cooper@santafe.edu (F. Cooper), avadh@lanl.gov (A. Saxena).

http://dx.doi.org/10.1016/j.physleta.2017.01.018
0375-9601/

for phenomenologically understanding meson spectroscopy as well as the electromagnetic spectroscopy of the positronium system. A good summary of this approach is found in the work of Crater, Becker, Wong and Van Alstine [20]. Our present paper, however, is only concerned with exploring the existence of bound state solutions of the two coupled $1+1$ dimensional NLDEs. We note that about four decades ago coupled NLD equations in $1+1$ dimensions were considered formally [21] but no explicit solutions were obtained.

Our strategy is to write the components of the two coupled nonlinear Dirac equations in rest frame solitary wave form, namely $\psi = e^{-i\omega_1 t}\{R_1\cos\theta, R_1\sin\theta\}$, $\phi = e^{-i\omega_2 t}\{R_2\cos\eta, R_2\sin\eta\}$ and then assume that the conservation law for linear momentum is satisfied independently for $i=1,2$. This assumption is equivalent to saying that $\theta(x)$, $\eta(x)$ have the *same* functional form they had when $g_3=0$. Once one makes that assumption we obtain an analytic expression for $R_i(x)$ which we then show approximately solves the differential equation for $R_i(x)$. The one situation which restricts the validity of this solution occurs in the scalar–scalar interaction case when one of the solitary wave solutions (when $g_3=0$) is of a double humped variety. In that case the solution is valid only when the dimensionless coupling constants g_3^2/g_2^2 and g_3^2/g_1^2 are $\le 1/100$. Otherwise the approximate analytic solutions we have found seem to be numerically accurate in both the scalar–scalar as well as the vector–vector coupled NLD equation as long as the two dimensionless constants are $\le 1/10$.

2. Scalar–scalar interactions

We are interested in solitary wave solutions of the coupled nonlinear Dirac equations (NLDEs) given by

$$(i\gamma^\mu\partial_\mu - m_1)\psi + g_1^2(\bar\psi\psi)\psi + g_3^2(\bar\phi\phi)\psi = 0\,, \tag{2.1}$$

$$(i\gamma^\mu\partial_\mu - m_2)\phi + g_2^2(\bar\phi\phi)\phi + g_3^2(\bar\psi\psi)\phi = 0\,. \tag{2.2}$$

We can eliminate one of the coupling constants by rescaling the fields, that is, if we let $\psi\to\psi/g_1$, $\phi\to\phi/g_2$, so that there are two independent dimensionless coupling constants

$$g_{32}^2 = g_3^2/g_2^2,\;\; g_{31}^2 = g_3^2/g_1^2, \tag{2.3}$$

as we will discover later. The field equations can be derived from the Lagrangian

$$\begin{aligned}\mathcal{L} &= \bar\psi(i\gamma^\mu\partial_\mu - m_1)\psi + \frac{g_1^2}{2}(\bar\psi\psi)^2\\ &+ \bar\phi(i\gamma^\mu\partial_\mu - m_2)\phi + \frac{g_2^2}{2}(\bar\phi\phi)^2 + g_3^2(\bar\psi\psi)(\bar\phi\phi)\\ &= \bar\psi(i\gamma^\mu\partial_\mu - m_1)\psi + \bar\phi(i\gamma^\mu\partial_\mu - m_2)\phi + \mathcal{L}_{int}.\end{aligned} \tag{2.4}$$

We notice the Lagrangian is symmetric under the interchange $\psi\to\phi$, $m_1\to m_2$ and $g_1\to g_2$.

We next choose the following representation of the γ matrices:

$$\gamma^0 = \sigma_3,\quad i\gamma_1 = \sigma_2, \tag{2.5}$$

where the σ_i are the usual Pauli spin matrices.

In the rest frame we assume that the two components of the solutions can be written as

$$\begin{aligned}\psi(x) &= \begin{pmatrix} A(x)\\ B(x)\end{pmatrix} e^{-i\omega_1 t} = R_1(x)\begin{pmatrix}\cos\theta(x)\\ \sin\theta(x)\end{pmatrix} e^{-i\omega_1 t},\\ \phi(x) &= \begin{pmatrix} C(x)\\ D(x)\end{pmatrix} e^{-i\omega_2 t} = R_2(x)\begin{pmatrix}\cos\eta(x)\\ \sin\eta(x)\end{pmatrix} e^{-i\omega_2 t}.\end{aligned} \tag{2.6}$$

In the absence of interactions ($g_3=0$), the solutions are of two types [19]. When $1>\omega/m>\omega_c/m$ then the solutions are single humped as they are always in the case of vector–vector interactions discussed below. However for the case $1>\omega_c/m>\omega/m$ the solutions are double humped and in that regime if the solutions when $g_3=0$ are of two different types, then we will find the approximate solutions we obtain are only valid for very small $g_{3i}^2 \le 1/100$. In component form these two coupled NLDEs can be written as

$$\begin{aligned}&\partial_x A + (m_1+\omega_1)B - g_1^2(A^2-B^2)B - g_3^2(C^2-D^2)B = 0,\\ &\partial_x B + (m_1-\omega_1)A - g_1^2(A^2-B^2)A - g_3^2(C^2-D^2)A = 0,\\ &\partial_x C + (m_2+\omega_2)D - g_2^2(C^2-D^2)D - g_3^2(A^2-B^2)D = 0,\\ &\partial_x D + (m_2-\omega_2)C - g_2^2(C^2-D^2)C - g_3^2(A^2-B^2)C = 0.\end{aligned} \tag{2.7}$$

These are symmetric under the interchange $\{A,B\}\to\{C,D\}$, $m_1\to m_2$, $\omega_1\to\omega_2$ and $g_1\to g_2$. These four equations can also be written if we let $y_i = R_i^2(x)$ as:

$$\begin{aligned}\frac{dy_1}{dx} &= 2[g_1^2 y_1^2(\cos 2\theta) + g_3^2 y_1 y_2(\cos 2\eta) - y_1 m_1]\sin 2\theta,\\ \frac{dy_2}{dx} &= 2[g_2^2 y_2^2(\cos 2\eta) + 2g_3^2 y_1 y_2(\cos 2\theta) - y_2 m_2]\sin 2\eta,\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\frac{d\theta}{dx} &= g_1^2 y_1\cos^2 2\theta + g_3^2 y_2\cos 2\theta\cos 2\eta - m_1\cos 2\theta + \omega_1,\\ \frac{d\eta}{dx} &= g_2^2 y_2\cos^2 2\eta + g_3^2 y_1\cos 2\theta\cos 2\eta - m_2\cos 2\eta + \omega_2.\end{aligned} \tag{2.9}$$

We can rewrite these equations in terms of the two dimensionless coupling constants by scaling $y_1\to y_1/g_1^2$, $y_2\to y_2/g_2^2$.

2.1. Conservation laws

We have that energy and momentum are conserved, namely

$$\partial_\mu T^{\mu\nu} = 0\,, \tag{2.10}$$

where the energy–momentum tensor is defined as

$$T_{\mu\nu} = i\bar\psi\gamma_\mu\partial_\nu\psi + i\bar\phi\gamma_\mu\partial_\nu\phi - g_{\mu\nu}\mathcal{L}\,, \tag{2.11}$$

and $\mathcal{L}$ is given by Eq. (2.4). From total momentum conservation, we find, just like for the single field NLDE, that for a solution that vanishes at $\pm\infty$ we have

$$T_{10} = \omega_1\bar\psi\gamma_1\psi + \omega_2\bar\phi\gamma_1\phi = 0 \tag{2.12}$$

and also

$$T_{11} = \omega_1\psi^\dagger\psi - m_1\bar\psi\psi + \omega_2\phi^\dagger\phi - m_2\bar\phi\phi + \mathcal{L}_{int} = 0\,. \tag{2.13}$$

Multiplying Eq. (2.1) on the left by $\bar\psi$ and Eq. (2.2) on the left by $\bar\phi$ and adding those two equations and then using Eq. (2.13) to eliminate the interaction terms of $\mathcal{L}_{int}$, we then obtain the equation:

$$\omega_1\psi^\dagger\psi - m_1\bar\psi\psi + i\bar\psi\gamma_1\partial_1\psi + \omega_2\phi^\dagger\phi - m_2\bar\phi\phi + i\bar\phi\gamma_1\partial_1\phi = 0\,, \tag{2.14}$$

which becomes using our ansatz

$$R_1^2\left(\frac{d\theta}{dx} + \omega_1 - m_1\cos 2\theta\right) + R_2^2\left(\frac{d\eta}{dx} + \omega_2 - m_2\cos 2\eta\right) = 0\,. \tag{2.15}$$

One also has that energy is conserved. The energy density is given by

$$T_{00}(x) = m_1\bar\psi\psi + m_2\bar\phi\phi = m_1 R_1^2\cos 2\theta + m_2 R_2^2\cos 2\eta\,, \tag{2.16}$$