Doctopic: Nonlinear science ARTICLE IN PRESS PLA:24093

Physics Letters A ••• (••••) •••–•••

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Non-periodic one-dimensional ideal conductors and integrable turbulence

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ARTICLE INFO

Article history:
Received 1 March 2016
Received in revised form 13 September 2016
Accepted 22 September 2016
Available online xxxx
Communicated by A.P. Fordy

Keywords:
Ideal conductors
Integrability
Solitonic gas

ABSTRACT

To relate the motion of a quantum particle to the properties of the potential is a fundamental problem of physics, which is far from being solved. Can a medium with a potential which is neither periodic nor quasi-periodic be a conductor? That question seems to have been never addressed, despite being both interesting and having practical importance. Here we propose a new approach to the spectral problem of the one-dimensional Schrödinger operator with a bounded potential. We construct a wide class of potentials having a spectrum consisting of the positive semiaxis and finitely many bands on the negative semiaxis. These potentials, which we call primitive, are reflectionless for positive energy and in general are neither periodic nor quasi-periodic. Moreover, they can be stochastic, and yet allow ballistic transport, and thus describe one-dimensional ideal conductors. Primitive potentials also generate a new class of solutions of the KdV hierarchy. Stochastic primitive potentials describe integrable turbulence, which is important for hydrodynamics and nonlinear optics. We construct the potentials by numerically solving a system of singular integral equations. We hypothesize that finite-gap potentials are a subclass of primitive potentials, and prove this in the case of one-gap potentials.

1. Introduction

Despite much work spanning almost 90 years, the evolution of a quantum particle in a one-dimensional bounded potential is far from being understood. Depending on the properties of the potential, there is a wide range of possibilities. Many random potentials (but not all, see [1]) display Anderson localization, meaning that the wave packet expands to a bounded size, and the particle does not move freely. In an opposite scenario the wave train propagates ballistically, and the particle can move to infinity in both directions (see [2]). This can happen, for example, in a periodic potential. A number of intermediate possibilities exist, for example the particle can diffuse to infinity, with the diffusion coefficient being a function of energy.

In this letter, we describe a large class of potentials that admit ballistic wave propagation. We give an effective analytic method for constructing such potentials and support this method with numerical computations.

The character of the evolution of a wave train is determined by the spectral properties of the Schrödinger operator

$$L\psi = (-\partial_x^2 + u(x))\psi = E\psi, \quad -\infty < x < \infty \tag{1}$$

A real number E belongs to the spectrum of L if (1) has one (non-degenerate) or two (doubly degenerate) linearly independent bounded solutions. The spectrum of a generic bounded potential can have a very complicated, fractal-like structure. Ballistic transport is possible for energies lying in an *allowed band*, in other words if there is an open interval such that the spectrum is doubly degenerate at each point of the interval.

In what follows we only consider potentials whose spectrum has such a band structure, consisting of a union of intervals on which it is doubly degenerate, separated by forbidden gaps. Periodic potentials, and certain quasi-periodic ones, have such a spectrum (see [3]). A generic periodic potential has infinitely many forbidden gaps, however, a dense subset of potentials has finitely many. Such finite-gap potentials play a fundamental role and can be explicitly described. A finite-gap potential is specified by choosing the gap boundaries on the real axis, a point inside each gap, and a choice of sign at each point. This data determines a hyperelliptic Riemann surface and a divisor on it, and the potential is explicitly given by the Matveev–Its formula in terms of the associated Riemann theta functions (see [4,5]). The resulting potential is quasi-periodic with $k \le N$ periods, and periodic potentials are obtained by imposing $N-1$ additional conditions.

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http://dx.doi.org/10.1016/j.physleta.2016.09.040
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Please cite this article in press as: D.V. Zakharov et al., Non-periodic one-dimensional ideal conductors and integrable turbulence, Phys. Lett. A (2016), http://dx.doi.org/10.1016/j.physleta.2016.09.040

Until recently it was believed that the algebro-geometric N-gap potentials are the only ones whose spectrum has a band structure with N gaps. In this letter, we show that this is not the case, and we effectively construct a much wider class of potentials that have such a spectrum. We call such potentials primitive. Unlike N-gap potentials, which are determined by finitely many parameters, primitive potentials are determined by an arbitrary continuous function. We hypothesize that all N-gap potentials are primitive, but prove this only for $N = 1$.

Unlike finite-gap potentials, primitive potentials are in general neither periodic nor quasi-periodic. Furthermore, numerical experiments show that they can be quite disordered, though we believe that they are not entirely random and have a hidden long-range order. We do not compute their correlation functions analytically, but numerical experiments indicate that these potentials are statistically almost uniform.

We believe that primitive potentials will have wide-ranging applications in diverse areas of physics. Let $u(x)$ be the potential of a one-dimensional medium consisting of irregularly spaced ions and a sea of non-interacting electrons. If Anderson localization holds, then the medium is an ideal dielectric. It seems natural to assume that a medium can be a conductor only if the potential function is periodic or quasiperiodic. We show that this is not the case, and that a much wider class of one-dimensional conductors is possible. If the potential is primitive, and the Fermi level is in one of the allowed bands, then such a medium is an ideal conductor, despite being non-periodic. This may help explain the conductivity of long non-periodic organic molecules, such as DNA.

Our results have important applications for a completely different area of physics. The Schrödinger equation is an auxiliary tool for integrating the Korteweg–de Vries (KdV) equation, which is one of the fundamental models of nonlinear wave dynamics. This procedure is known as the inverse spectral transform, or IST, discovered in 1967 in [7]. Under the IST, the potential is assumed to be time-dependent, and it turns out that KdV evolution does not change the spectrum of the associated stationary Schrödinger operator. Moreover, primitive potentials remain primitive. Hence, our method also constructs a new family of exact solutions of KdV, and the higher KdV hierarchy, which are bounded but non-vanishing as $|x| \to \infty$. Computer simulations show that these solutions are quite irregular.

Integrable nonlinear wave equations, such as KdV, describe a number of important physical systems: waves on shallow water, nonlinear waves in optic fibers, and so on. All of these systems are in need of a statistical description. The first steps in such a theory, known as integrable turbulence [8], have already been made.

We note that, although this letter describes a somewhat complicated mathematical theory, we state most propositions without proof, and we plan to publish them elsewhere. Our method also includes an intricate numerical algorithm, using multiscale accuracy, the details of which will also be published separately.

2. Primitive potentials

We give a construction of a wide class of potentials whose spectrum consists of the positive semiaxis and N allowed bands on the negative semiaxis.

Primitive potentials are the continuous limits of reflectionless Bargmann potentials [9], which are also fixed-time slices of N-soliton solutions of the KdV hierarchy. We omit the details of this limiting transition and give a direct construction using the dressing method, following [10]. We consider a distribution $T(k)$ on the complex k-plane, which we call the dressing function, satisfying the following conditions:

$$T(\bar{k}) = -\overline{T(-k)}, \qquad \int |T(k)| dk \wedge d\bar{k} < \infty, \tag{2}$$

here and now on we assume integration over the entire complex plane unless explicitly specified otherwise. We consider the following integral equation on a function $\chi(x,k)$ defined on the complex k-plane (in what follows we write $T(k)$ and $\chi(x,k)$ without assuming either to be analytic):

$$\chi(x,k) = 1 - \frac{1}{2\pi}\int \frac{T(-q)\chi(x,q)e^{-2iqx}}{k+q} dq \wedge d\bar{q}, \tag{3}$$

where $\overline{\chi(x,-k)} = \chi(x,\bar{k})$ and x is a parameter.

Suppose that the dressing function is such that the equation (3) has for all x in an interval (x_1, x_2) a unique solution satisfying $\chi \to 1$ as $|k| \to \infty$. Then the function χ has the following asymptotic expansion:

$$\chi(x,k) = 1 + \frac{i\chi_0(x)}{k} + \cdots$$

The function $\chi_0(x)$ is real-valued by virtue of (2)–(3). Furthermore, $\chi(x,k)$ is a solution of the equation:

$$\chi_{xx} - 2ik\chi_x - u(x)\chi = 0, \quad u(x) = 2\frac{d}{dx}\chi_0(x),$$

and the function $\psi = \chi e^{ikx}$ is a solution of the Schrödinger equation (1) with $E = k^2$. This, of course, does not mean that E is a point of the spectrum. For this to hold, the following conditions need to be satisfied:

1. Equation (3) must have a solution all x, i.e. $x_1 = -\infty$ and $x_2 = +\infty$. Otherwise, at the boundaries $u(x)$ will have a singularity (generically a pole of order two).
2. The potential $u(x)$ must be bounded for all x.
3. At least one solution of the Schrödinger equation must be bounded for all x.

The first two of these conditions impose strong restrictions on the dressing function $T(k)$. We choose the dressing function in the following way. Let $0 < k_1 < k_2$, and let $R_1(\kappa)$ and $R_2(\kappa)$ be two real-valued functions on $[k_1, k_2]$, which we extend by zero to the entire real axis. Let $k = k_R + ik_I$, and define

$$T(k) = i\delta(k_R)[R_1(k_I) - R_2(-k_I)], \tag{4}$$

where $\delta(k_R)$ is the one-dimensional Dirac delta function. The symmetry conditions from (2) follow. A function $\chi(x,k)$ satisfying (3) with such a $T(k)$ is analytic on the k-plane away from two cuts $k_1 < \operatorname{Im} k < k_2$ and $-k_2 < \operatorname{Im} k < -k_1$ on the imaginary axis. It has the following representation:

$$\chi(x,k) = 1 + i\int_{k_1}^{k_2} \frac{\varphi(x,q)e^{-qx}}{k - iq} dq + i\int_{k_1}^{k_2} \frac{\psi(x,q)e^{qx}}{k + iq} dq. \tag{5}$$

Substituting (4) and (5) into (3) gives a system of singular integral equations on φ and ψ. These equations are equivalent a vector Riemann–Hilbert problem. Denote $\Xi(k) = [\chi(k) \ \ \chi(-k)]^T$, and let Ξ^+ and Ξ^- be the right and left values of Ξ on the cuts. Then the problem is

$$\Xi^+(i\kappa) = M(\kappa)\Xi^-(i\kappa), \quad \Xi^+(-i\kappa) = M^T(\kappa)\Xi^-(-i\kappa) \tag{6}$$

for $\kappa \in [k_1, k_2]$, where the transition matrix is

$$M(x,\kappa) = \frac{1}{1 + R_1R_2}\begin{bmatrix} 1 - R_1R_2 & 2iR_1e^{-2\kappa x} \\ 2iR_2e^{2\kappa x} & 1 - R_1R_2 \end{bmatrix}$$

We claim that if R_1 and R_2 are non-negative functions satisfying the Hölder condition for some $\alpha > 0$, then this Riemann–Hilbert problem has a unique solution for all x with normalization