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# Performing derivative and integral operations for optical waves with optical metamaterials ☆

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ABSTRACT

The graded refractive index waveguides can perform Fourier transform for an optical wave. According to this characteristic, simpler optical metamaterials with three waveguides are theoretically proposed, in which all of the waveguides are materials with a positive refractive index. By selecting the appropriate refractive index and structure size, the theory and simulations demonstrated that these metamaterials can perform mathematical operations for the outline of incident optical waves, including the first-order derivative, second-order derivative and the integral.





## 1. Introduction

Metamaterials [1,2] are a recent type of artificial structure materials, and some physical properties presented by metamaterials are related to the physical size of their structure. Metamaterials provide a brand new concept and method for creating many materials with special physical properties in a flexible way. Crystal would be the most intuitive example, which inherently boasts a significantly orderly structure. Its order is mainly due to the arrangement of atoms which endows the crystal materials with specific physical properties that are not possessed by the same materials when disordered. Consequently, people come to realize the expected physical characteristics through the ordered structure at all levels, and thus obtain physical properties that are not possessed by the disordered or unorganized materials at the same level.

Currently, studies on metamaterials have mainly concentrated on phototonic crystals [3–5], zero refractive index waveguides [6,7], materials with high refractive index [8–10], strong anisotropic materials [11–14], and left-handed materials [15,16], among others. Metamaterials with mathematical operation functions have come to the fore in research in recent years. In 2014, Cui [17] et al. proposed the concept of coding metamaterials, digital metamaterials and programming metamaterials. Also, an electromagnetic wave is manipulated with different coding sequences of the metamaterials grain, which consists of “0” and “1”. Moreover, Silva et al. [18] pioneered the creation of metamaterials analog computing, using a metasurface design in which the two ends of the materials are different. The left is normal material with positive permittivity and positive permeability, while the right is a dual-negative material with negative permittivity and negative permeability. In this paper, the physical properties of the materials are described with respect to the refractive index. All the materials have a positive refractive index and can perform the same functions for optical waves.

## 2. Theory

When optical waves propagate through the Graded Refractive Index (GRIN) waveguides, the GRIN waveguides perform one Fourier transform [19] at each characteristic distance $Lg$, and the refractive index of GRIN satisfies $\eta(y) = \eta_1\sqrt{1-(\eta_2/\eta_1)y^2}$, in which, $y$ is $y$-axis coordinate in two-dimensional Cartesian coordinate system, and $\eta_1$ is the refractive index at $y = 0$, $\eta_2/\eta_1 = [\pi/(2Lg)]^2$. More details can be found in Ref. [19]. Supposing that there are three waveguides (1, 2 and 3) that nest a metamaterial, waveguides 1 and 3 are GRIN materials with $\eta_1(y) = -\eta_3(y) = \eta(y)$, which perform Fourier and inverse Fourier transform, respectively, and waveguide 2 is the kernel transform function and named as G(y). When z-component of the electric field $E_z(y)$ is

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inputted to the waveguide 1, the output function becomes $f'(k_y) = \mathcal{F}[E_z(y)]$, with $\mathcal{F}(.)$ denoting Fourier transform. The Fourier variable $k_y$ is in the same domain of $y$, so $y$ plays the role of $k_y$. And the output function of waveguide 2 becomes $G(y)\mathcal{F}[E_z(y)]$. Since waveguide 3 performs inverse Fourier transform, the final output function is $\mathcal{F}^{-1}\{G(y)\mathcal{F}[E_z(y)]\}$. Now, in order to realize the derivative operations, there must be

$$\frac{d^n E_z(y)}{dy^n} = \mathcal{F}^{-1}\{G(y)\mathcal{F}[E_z(y)]\} \tag{1}$$

taking Fourier transform on both sides, we have

$$\mathcal{F}\left[\frac{d^n E_z(y)}{dy^n}\right] = G(y)\mathcal{F}[E_z(y)] \tag{2}$$

according to the properties of the Fourier function $\mathcal{F}[f^{(n)}(t)] = (i\omega)^n \mathbf{F}(\omega)$, we obtain

$$\mathcal{F}\left[\frac{d^n E_z(y)}{dy^n}\right] = (ik_y)^n f'(k_y) = (ik_y)^n \mathcal{F}[E_z(y)] \tag{3}$$

from Eqs. (2)–(3), we can obtain our transfer function $G(y) = (ik_y)^n \propto (-iy)^n$. Supposing that the dimension of waveguide is finite and limited by $W$, the normalized transform function becomes $G(y) \propto (-iy/y_0)^n$, where $y_0 = W/2$.

When plane optical waves propagate along the positive direction of the x axis, the electric field satisfies the Helmholtz equation

$$\frac{d^2}{dx^2}\boldsymbol{E}(x) + k^2\boldsymbol{E}(x) = 0 \tag{4}$$

in which, $\boldsymbol{E}(x)$ is the electric field, $k = (2\pi/\lambda_0)\sqrt{\epsilon_r\mu_r}$, $\lambda_0$ is the wavelength of incident waves, and $\epsilon_r$ and $\mu_r$ are the relative permittivity and relative permeability, respectively. Since $\epsilon_r\mu_r = (\eta - i\kappa)^2$ [20], where $\eta$ and $\kappa$ are the real and imaginary parts of the refractive index, respectively, $k$ can also be expressed as $k = (2\pi/\lambda_0)(\eta - i\kappa)$. Since electric field satisfies the Helmholtz equation, so in order to perform the derivative and integral operations for optical waves, the refractive index of waveguide 2 needs to be determined. One solution of Eq. (4) is

$$\boldsymbol{E}(x) = \boldsymbol{E}_0 e^{ikx} \tag{5}$$

where $E_0$ is the electric field at $x = 0$. Since the electric field on the left side of waveguide 2 is $\mathcal{F}[E_z(y)]$, likewise, after propagation through $\Delta$, the electric field on the right side of waveguide 2 is $\mathcal{F}[E_z(y)]e^{ik\Delta}$ with $\Delta$ being the thickness of medium 2. As $\mathcal{F}[E_z(y)]e^{ik\Delta} = G(y)\mathcal{F}[E_z(y)] \propto (-iy/y_0)^n\mathcal{F}[E_z(y)]$, we obtain

$$e^{ik\Delta} = G(y) \propto (-iy/y_0)^n = \left(-i\frac{2y}{W}\right)^n \tag{6}$$

namely

$$e^{ik\Delta} = e^{i\frac{2\pi}{\lambda_0}[\eta_2(y) - i\kappa_2(y)]\Delta} \propto \left(-i\frac{2y}{W}\right)^n \tag{7}$$

In order to produce an output profile proportional to the first derivative of the input function, taking $G(y) = (-iy/y_0)$, according to Eq. (7), we have

$$\eta_2(y) - i\kappa_2(y) = i\frac{\lambda_0}{2\pi\triangle}\log\left(i\frac{W}{2y}\right) \tag{8}$$

As for the second derivative, taking $G(y) = (-iy/y_0)^2$, the refractive index is

$$\eta_2(y) - i\kappa_2(y) = i\frac{2\lambda_0}{2\pi\triangle}\log\left(i\frac{W}{2y}\right) \tag{9}$$

namely

$$\eta_2(y) = \mathrm{Re}\left[i\frac{n\lambda_0}{2\pi\triangle}\left(\frac{\pi}{2}i + \log\frac{W}{2y}\right)\right] \tag{10}$$

$$\kappa_2(y) = -\mathrm{Im}\left[i\frac{n\lambda_0}{2\pi\triangle}\left(\frac{\pi}{2}i + \log\frac{W}{2y}\right)\right] \tag{11}$$

In this context, $n$ is 1 and 2 in Eqs. (10)–(11), and the system can complete the first-order derivative and second-order derivative operations for optical waves, respectively.

For the purposes of performing integral operations, the transfer function is $G(y) = (-iy/d)^{-1}$, where $d$ is any normalizing length, in this case $d = \lambda_0/4$. In order to avoid adapting to requirements for transmission coefficients with magnitude larger than unity for $|y| < d$, we truncate the refractive index at $|y| = d$, and assume that the absolute value of the refractive index is constant within this range. Therefore, the refractive index functions for integral are

$$\eta_2(y) - i\kappa_2(y) = i\frac{\lambda_0}{2\pi\triangle}\log\left(-i\frac{y}{d}\right) \quad \text{for} \quad |y| > d \tag{12}$$

$$\eta_2(y) - i\kappa_2(y) = -\frac{\lambda_0}{4\triangle}\mathrm{sign}\left(\frac{y}{d}\right) \quad \text{for} \quad |y| < d \tag{13}$$

The above results indicate that if the waveguides 1 and 3 are GRIN materials, and the refractive index of waveguide 2 satisfies Eqs. (8)–(9) and Eqs. (12)–(13), respectively, the whole structure will perform the derivative and integral operations accordingly. However, it must also be noted that waveguide 2 and 3 also have negative refractive index, thereby increasing the difficulty of the experiment, the solution to this problem is as follows:

Since $\eta - i\kappa = \pm\sqrt{\epsilon_r\mu_r}$, so Eq. (4) also can be expressed as

$$\frac{d^2}{dx^2}\boldsymbol{E}(x) + \left(\frac{2\pi}{\lambda_0}\right)^2(\eta - ik)^2\boldsymbol{E}(x) = 0 \tag{14}$$

As for $\eta - i\kappa = \pm\sqrt{\epsilon_r\mu_r}$, when $\epsilon_r$ and $\mu_r$ are positive, the refractive index exhibits positive value, and when $\epsilon_r$ and $\mu_r$ are negative, the refractive index exhibits negative value [21,22]. Eq. (14) implies that regardless of the refractive index being negative or positive, there is no difference in Helmholtz equation. Since positive refractive index produces the same effect as negative refractive index, the above derivative and integral operations can be performed with positive or negative refractive index.

## 3. Modeling

According to the above theory, the metamaterials with three waveguides are proposed. The basic model is shown in Fig. 1, which presents a modeling diagram of COMSOL Multiphysics 4.4, Electromagnetic Waves, and Frequency Domain. The length and width of the structure are $2Lg + \Delta$ and $W$, respectively. The tops and bottoms of the waveguides are perfect matching layer with thickness $d_1 = \lambda_0/4$. Limited by memory and speed of the computer, the calculation area can not extend to infinity, so the boundary is truncated by boundary conditions. The outer boundary condition is defined as the scattering boundary, and the boundary conditions on the boundaries between the waveguides 1 and 2 and between 2 and 3 are continuous boundary. And the optical waves enter from the left and exit from the right.

The structural parameters are $\lambda_0 = 0.5\mu m$, $W = 9.87637\lambda_0$, $Lg = 11.61928\lambda_0$, and $\Delta = \lambda_0/3.01231$. The z-component of the electric field of incident optical waves varies with $y$, and the specific expression is $E_z(y) = aye^{-y^2/b}$ (in which $a = 2.1/\lambda_0$, $b = \lambda_0^2/0.9$) or $E_z(y) = ce^{-y^2/d_2^2}(4y^2/d_2^2 - 2)$ (in which $c = 0.5$,